SOME PROBLEMS IN APPROXIMATION THEORY

By

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This dissertation is dedicated to the memories of

Professor Arun Kumar Varma

and

Robert Patrick McGrath

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Abstract of Dissertation Presented to the Graduate School of the University of Florida in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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We begin by providing an historical background and some results concerning polynomial approximation and interpolation. Next we consider Birkhoff, or lacunary, interpolation and its development. Then we provide the historical basis and development of Markov-type inequalities, and related best constant problems when the class of polynomials is restricted in some way.

We first investigate the $(0,1,3,4)$ case of Birkhoff interpolation where the nodes of interpolation are the zeros of the integral of the Legendre polynomial. We prove the existence and uniqueness in the 'modified' $(0,1,3,4)$ case, and then provide an explicit representation in this case. Next we prove that the 'modified' $(0,1,3,4)$ Birkhoff-Fejér operator (based on the zeros of the integral of the Legendre polynomial) converges uniformly for the entire class of continuous functions on $[-1,1]$. This provides only the second known case of such a Birkhoff-Fejér operator – the first being the $(0,3)$ case, both 'modified' and 'pure', studied first by Akhlaghi, Chak

and A. Sharma who proved the existence and uniqueness and provided explicit forms for the fundamental polynomials, and then by J. Szabados and A.K. Varma who provided a new representation for the fundamental polynomials of the first kind, and proved the convergence results.

Let L_n denote the Lorentz class of nonnegative polynomials of degree n on $[-1,1]$. In 1940, P. Erdös proved a refinement of Markov's inequality for polynomials with all real zeros which are outside $(-1,1)$. We extend the results of P. Erdös, P. Erdös and A. K. Varma, and G. V. Milovanović and M. S. Petković for $P_n \in L_n$ in the L^2 norm with the ultraspherical weight $w(x) = (1-x^2)^\alpha$, $\alpha > -1$, and we extend these results in a weighted L^4 norm.

Let now H_n be the set of all polynomials of degree n whose zeros are all real and lie inside $[-1,1]$. We provide the lower bound analogues to the Erdös-type inequalities for $P_n \in H_n$, as well as extend the results of P. Turán and A. K. Varma with an asymptotically sharp result in the L^p norm for p an even integer.

We conclude with a summary of the results and note some related open problems.

CHAPTER 1
INTRODUCTION

1.1 Approximation by Polynomials

In 1715, the English mathematician Brook Taylor (1685-1731) published his generalization of the Mean Value Theorem [46]. His method approximates a given n–times differentiable function f by a polynomial P_n of degree n in $(x-a)$ required to satisfy the conditions

(1.1.1)

$$P_n^{(k)}(a) = f^{(k)}(a), \quad k = 0, \ldots, n.$$

These conditions yield the polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

the so-called n^{th} Taylor polynomial. Taylor used these polynomials to approximate solutions to equations.

Let us denote by $C[a,b]$ the class of continuous functions on the interval $[a,b]$. Later on we shall have use for the big 'O' notation, whereby $O(k)$ means less than or equal to a positive constant times k. Let us denote by Π_n the class of all algebraic polynomials of degree at most n, and denote the uniform norm for $f \in C[-1,1]$

$$||f|| = \max_{-1 \leq x \leq 1} |f(x)|.$$

Lagrange proved the following version of Taylor's Theorem with remainder.

Theorem 1.1 If f and its first $n+1$ derivatives are continuous on an open interval (c,d) and if x and a are points of (c,d), then

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\theta_{n+1})}{(n+1)!}(x-a)^{n+1},$$

where θ_{n+1} is some number between x and a.

Unfortunately, the Taylor polynomials require a function f to be n–times differentiable, and still may yield poor approximations to f outside a very small neighborhood of a, as some derivatives of f at a may be very large compared to $f(a)$. Also, they are not a very efficient way to approximate a function. For example, the error in the approximation for $f(x) = e^x$ by the third Taylor polynomial P_3 about $a = 0$ on the interval $[-1, 1]$ can be seen to be

$$||e^x - P_3(x)|| \approx 0.0516,$$

where the error is not evenly distributed through the interval. As is typical of approximations by Taylor polynomials, the error is much smaller near the origin than near the endpoints ± 1.

In 1885, K. Weierstrass [55] discovered a theorem that founded a theory of the approximation of functions, which can be stated as follows.

Theorem 1.2 If f is in $C[-1, 1]$, then there exists a sequence of polynomials P_n such that $P_n \to f$ uniformly on $[-1, 1]$.

The so-called Bernstein polynomials

$$B_n(f, x) = \sum_{k=0}^{n} f\Big(\frac{2k-n}{n}\Big)\binom{n}{k}\frac{1}{2^n}(1+x)^k(1-x)^{n-k}, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!},$$

provide such a sequence. Observe that if $f \geq 0$, then $B_n(f, x) \geq 0$. Thus, we say that B_n is a positive operator. It turns out that if $f \geq 0$, then B_n is an element of the Lorentz class of polynomials, which we shall investigate further in Chapter 4. The Bernstein polynomials also have the nice property that for any given r–times continuously differentiable function f, we have $B_n^{(j)}(f, x) \to f^{(j)}(x)$ $(j = 0, \ldots, r)$ uniformly on $[-1, 1]$. On the other hand, the convergence of the Bernstein polynomials

is generally very slow. For example, if we choose $f(x) = x^2$, then

$$\lim_{n\to\infty} n[B_n(f,x) - f(x)] = x(1-x),$$

so that for large values of n we have

$$||B_n(f,x) - x^2|| = O\left(\frac{1}{n}x(1-x)\right).$$

Thus, the error does not decrease rapidly with n, even though f is a very simple function.

Let us digress for a moment to consider how Bernstein may have come up with these polynomials. Bernstein knew probability theory, and probably reasoned as follows. First, suppose that the probability of an event occurring is x, where $0 \le x \le 1$, and so the probability of the event not occurring is $1-x$. Now, the probability of the event occurring precisely k times in n attempts is given by $\binom{n}{k} x^k(1-x)^{n-k}$. It follows then that

$$\sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} = \sum_{k=0}^{n} \text{(sum of probabilities)} = 1.$$

One may also observe

$$1 = [(1-x)+x]^n = \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k}.$$

Suppose now $f \in C[0,1]$ and x is chosen randomly in $[0,1]$. For a given positive integer n, consider the set $\{f(\frac{k}{n}) : k = 0, \ldots, n\}$. If n is large enough, then at least one of the numbers $f\left(\frac{k}{n}\right)$ lies close to $f(x)$. We want to find a weighted sum

$$\sum_{k=0}^{n} w(k,x) f\left(\frac{k}{n}\right), \quad \sum_{k=0}^{n} w(k,x) = 1,$$

that yields a good approximation to $f(x)$. It follows from the Law of Large Numbers in probability theory that choosing the weights $w(k,x) = \binom{n}{k} x^k(1-x)^{n-k}$, the polynomials

$$\sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k(1-x)^{n-k}$$

converge uniformly to the function f on the interval $[0,1]$. These are simply the Bernstein polynomials on $[0,1]$. Once Bernstein found these polynomials, he gave a proof of uniform convergence without using the Law of Large Numbers.

As a tool to measure the rate of convergence, we now introduce the classical modulus of smoothness (or continuity) of order s

$$\omega_s(f,\delta) = \sup_{0<h\leq\frac{\delta}{2}} |\Delta_h^s f|,$$

where

$$\Delta_h^1 f(x) = f(x+h) - f(x-h),$$

$$\Delta_h^s f(x) = \Delta_h^{s-1} f(x+h) - \Delta_h^{s-1} f(x-h).$$

It has been shown by G.G. Lorentz [21] that the error in the case of the Bernstein polynomials is

$$||B_n(f) - f|| = O\left(\omega_1\left(f, \frac{1}{\sqrt{n}}\right)\right),$$

and this is the best possible.

As $C[-1,1]$ is a normed linear space, for a given $f \in C[-1,1]$, there always exists a polynomial of best approximation to f. In fact, for each n there exists a unique polynomial $p_n^* \in \Pi_n$ such that

$$||f - p_n^*|| \leq ||f - p|| \text{ for all } p \in \Pi_n.$$

Set $E_n(f) = ||f - p_n^*||$. It has been shown by D. Jackson [19] that $E_n(f) \leq 6\omega_1\left(f, \frac{1}{n}\right)$. Because it is in general very difficult to obtain the polynomial of best approximation p_n^*, one often considers the best least squares approximation. That is, one minimizes $||f - p_n||_{L^2}$ where

$$||f||_{L^2} = \left(\int_{-1}^{1} \omega(x)(f(x))^2 dx\right)^{\frac{1}{2}},$$

and where $\omega(x) \geq 0$ and $\int_{-1}^{1} \omega(x)dx$ exists. These approximations are usually pretty easy to compute, typically yielding a good uniform approximation to f, superior

to that given by the Taylor polynomials. For example, we have for the function $f(x) = e^x$ that the least squares approximation $r_3^*(x)$ ($\omega(x) \equiv 1$) of degree 3 gives the error

$$||e^x - r_3^*(x)|| \approx 0.0112.$$

The error for the polynomial of best approximation $p_3^*(x)$ of degree 3 here is

$$||e^x - p_3^*(x)|| \approx 0.0055.$$

The weight function $\omega(x)$ allows for different levels of importance to be given to the error at different points in the interval. Later, in Chapters 4 and 5, we shall investigate some inequalities for polynomials in such a weighted L^2 norm.

1.2 Lagrange and Hermite-Fejér Interpolation

Let us now consider the following problem. Given n distinct nodes

$$-1 \leq x_n < x_{n-1} < \cdots < x_1 \leq 1,$$

and data $y_1, y_2, \ldots, y_n$, find an algebraic polynomial of least degree whose graph passes through the points (x_k, y_k) for $k = 1, \ldots, n$. That is, find a polynomial, say L_{n-1}, such that

(1.2.1)

$$L_{n-1}(x_k) = y_k, \quad k = 1, \ldots, n.$$

We denote the Kronecker delta function $\delta_{k\nu}$ by

$$\delta_{k\nu} = \begin{cases} 1, & \text{if } \nu = k, \\ 0, & \text{if } \nu \neq k. \end{cases}$$

Now, if we could find polynomials $\ell_\nu(x)$ ($\nu = 1, \ldots, n$) such that

(1.2.2)

$$\ell_\nu(x_k) = \delta_{k\nu}, \quad k = 1, \ldots, n,$$

then we could write

(1.2.3)
$$L_{n-1}(x) = \sum_{\nu=1}^{n} y_\nu \ell_\nu(x),$$

and the polynomial L_{n-1} satisfies (1.2.1). But,

$$\ell_\nu(x) = \frac{(x-x_1)\cdots(x-x_{\nu-1})(x-x_{\nu+1})\cdots(x-x_n)}{(x_\nu-x_1)\cdots(x_\nu-x_{\nu-1})(x_\nu-x_{\nu+1})\cdots(x_\nu-x_n)} \in \Pi_{n-1}$$

is a polynomial satisfying (1.2.2). Thus, we have shown the existence of a polynomial $L_{n-1} \in \Pi_{n-1}$ satisfying the conditions (1.2.1).

Suppose now that there exists another polynomial $p_{n-1} \in \Pi_{n-1}$ that also satisfies (1.2.1). Then

$$L_{n-1}(x_k) - p_{n-1}(x_k) = 0, \quad k = 1, \ldots, n,$$

and $L_{n-1}(x) - p_{n-1}(x)$ is a polynomial of degree at most $n-1$ having n zeros. Then $L_{n-1}(x) - p_{n-1}(x) \equiv 0$, or equivalently, $L_{n-1} \equiv p_{n-1}$. Thus, there always exists a unique polynomial of degree at most $n-1$ satisfying (1.2.1), and it is given by (1.2.3). We have just derived the so-called Lagrange interpolation formula. The polynomials $\ell_\nu(x)$ are called the fundamental polynomials of Lagrange interpolation. Lagrange was interested in using interpolation to exploit the information in astronomical tables, and around 1790 he presented a paper to the Academy of Sciences in Berlin.

If we define $\omega(x) = (x-x_1)(x-x_2)\cdots(x-x_n)$, then the numerator of $\ell_\nu(x)$ can be written as $\frac{\omega(x)}{(x-x_\nu)}$. Observe then that

$$\omega'(x_\nu) = (x_\nu - x_1)\cdots(x_\nu - x_{\nu-1})(x_\nu - x_{\nu+1})\cdots(x_\nu - x_n)$$

is the denominator of $\ell_\nu(x)$. That is, we can write

$$\ell_\nu(x) = \frac{\omega(x)}{(x-x_\nu)\omega'(x_\nu)}.$$

This is a more useful form of the fundamental polynomials of Lagrange interpolation. Concerning the error in Lagrange interpolation, Cauchy showed that for a function f that is n-times differentiable, we have

$$f(x) - L_{n-1}(x) = \frac{\omega(x)}{n!} f^{(n)}(\xi),$$

for some number $\xi \in (-1,1)$. Notice that if $f \in \Pi_{n-1}$, then $f^{(n)} \equiv 0$, and we have $L_{n-1} \equiv f$. We say then that L_{n-1} is a projection operator.

One might think it reasonable to expect that if a system of equally spaced nodes are chosen, then for a continuous function f on the interval $[-1,1]$ we should have $L_{n-1} \to f$ uniformly on $[-1,1]$. In 1901, C. Runge [30] presented his classical example that this is not necessarily the case. Let $x_k = -1 + \frac{2(k-1)}{n-1}$ for $k = 1, \ldots, n$, and choose $f(x) = \frac{1}{1+25x^2}$. Runge showed that the Lagrange interpolation polynomial does not converge to the continuous function f on $[-1,1]$. In fact, he showed that for $0.72 < |x| < 1$, we have

$$\lim_{n\to\infty} |L_{n-1}(f,x)| = \infty.$$

This result is rather disappointing, and in fact in 1914, Faber [13] published his result that for every choice of nodes

$$\sum_{\nu=1}^{n} |\ell_\nu(x)| > \frac{\log n}{8\sqrt{\pi}}.$$

Unfortunately, this means that for any system of nodes, one can always find a function f such that $L_{n-1}(f)$ becomes unbounded. That is, there is no universally effective system of nodes. A proof of this result can be found in the book of Rivlin [29]. Let us define now the Lebesgue constant

$$\Lambda_{n-1} = \max_{-1\le x\le 1} \sum_{\nu=1}^{n} |\ell_\nu(x)|.$$

It is easy to see that

$$||f - L_{n-1}|| \le E_{n-1}(f)(1+\Lambda_{n-1}) \le 6(1+\Lambda_{n-1})\omega_1\Big(f, \frac{1}{n}\Big).$$

Thus, for given $f \in C[-1,1]$ and a given system of nodes, $L_{n-1}(f,x) \to f(x)$ uniformly if $\Lambda_{n-1}\omega_1\left(f, \frac{1}{n}\right) \to 0$.

In looking for interpolation polynomials which are uniformly convergent for the whole class $C[-1,1]$, L. Fejér [14] considered the so-called Hermite [17] interpolation polynomial $H_{2n-1} \in \Pi_{2n-1}$. Here, given data $y_1, \ldots, y_n$ and $y'_1, \ldots, y'_n$, we require that H_{2n-1} satisfies the following conditions

$$H_{2n-1}(x_k) = y_k, \quad H'_{2n-1}(x_k) = y'_k, \quad k = 1, \ldots, n.$$

Thus, on any set of real distinct nodes, the Hermite interpolation polynomial has the form

$$H_{2n-1}(x) = \sum_{k=1}^{n} y_k A_k(x) + \sum_{k=1}^{n} y'_k B_k(x)$$

where the fundamental polynomials are given by

$$A_k(x) = \ell_k^2(x)[1 - 2\ell'_k(x_k)(x - x_k)], \quad B_k(x) = (x - x_k)\ell_k^2(x), \quad k = 1, \ldots, n.$$

In the case that the nodes are the zeros of the Chebyshev polynomial $T_n(x)$, we have that

$$A_k(x) = \ell_k^2(x)\left(\frac{1 - x x_k}{1 - x_k^2}\right) \geq 0,$$

and it follows that

$$\sum_{k=1}^{n} |A_k(x)| = 1.$$

Fejér showed that for any continuous function f on $[-1,1]$, the operator

$$R_{2n-1}(f,x) = \sum_{k=1}^{n} f(x_k) A_k(x)$$

converges uniformly to f on $[-1,1]$. Two proofs of this result, as well as a nice discussion, are given in the paper of T.M. Mills [27]. Note that this operator has the properties that

$$R_{2n-1}(f,x_k) = f(x_k) \text{ and } R'_{2n-1}(f,x_k) = 0, \quad k = 1, \ldots, n.$$

Thus, when all the higher derivative information is set equal to 0, we refer to such a polynomial as a Hermite–Fejér operator. This was one of the first interpolatory proofs of the Weierstrass Approximation Theorem. It has been shown that for $f \in C[-1,1]$ we have

$$||H_{2n-1}(f) - f|| = O\left(\omega_1\left(f, \frac{\log n}{n}\right)\right),$$

and this is the best possible in the sense that for the function $g(x) = |x|$ we have

$$||H_{2n-1}(g) - g|| \geq c\left(\omega_1\left(g, \frac{\log n}{n}\right)\right).$$

One can generalize the Lagrange and Hermite interpolation problem to the so–called general Hermite interpolation problem (or simply Hermite interpolation). In this problem we seek a polynomial H_{nm-1} satisfying the mn conditions

$$H^{(j)}_{nm-1}(x_k) = y^{(j)}_k, \quad j = 0, \ldots, m-1, \quad k = 1, \ldots, n,$$

where the numbers $y^{(j)}_k$ are given data. Notice that when $n = 1$, H_{nm-1} is just the $(m-1)^{st}$ Taylor polynomial. The $(0, 1, \ldots, m-1)$ Hermite interpolation problem always has a unique solution $H_{nm-1} \in \Pi_{nm-1}$ on any set of real distinct nodes.

Despite the positive result in the $(0,1)$ case of Hermite interpolation, J. Szabados and A.K. Varma [42] showed that the Lebesgue constant in the $(0,1,2)$ case for every choice of nodes has the property

$$\Lambda_{3n-1} = \max_{-1\leq x\leq 1} \sum_{k=1}^{n} |A_{0,3,k}(x)| > c \log n,$$

where the $A_{0,3,k}(x)$ are the fundamental polynomials of the first kind of (0,1,2) interpolation. Thus, one cannot obtain uniform convergence for the whole class $C[-1,1]$ for the $(0,1,2)$ Hermite–Fejér operator for any choice of nodes.

Later, J. Szabados [40] showed that

$$H_{nm-1}(f, x) = \sum_{k=1}^{n} \sum_{j=0}^{m-1} y^{(j)}_k A_{jk}(x),$$

where the fundamental polynomials of the $(j+1)^{st}$ kind $A_{jk}(x)$ of $(0,1,\ldots,m-1)$ Hermite interpolation are given by

$$A_{jk}(x) = \frac{(\ell_k(x))^m}{j!}\sum_{i=0}^{m-j-1}\frac{[\ell_k^{-m}(x)]_{x_k}^{(i)}}{i!}(x-x_k)^{i+j}, \quad j=0,\ldots,m-1 \quad k=1,\ldots,n,$$

and he showed that for every choice of nodes

$$\left|\left|\sum_{k=1}^{n}|A_{j,k}(x)|\right|\right| \geq \begin{cases} c_1\left(\frac{\log n}{n^j}\right), & \text{if } m-j \text{ is odd,} \\ \frac{c_2}{n^j}, & \text{if } m-j \text{ is even.} \end{cases}$$

In particular, when m is odd, we have for any system of nodes

$$\left|\left|\sum_{k=1}^{n}|A_{0,k}(x)|\right|\right| \geq c\log n$$

so that there cannot be uniform convergence for the whole class $C[-1,1]$ for the $(0,1,\ldots,m-1)$ Hermite-Fejér operator (m odd) for any choice of nodes.

1.3 Birkhoff and Birkhoff-Fejér Interpolation

In 1906, G. D. Birkhoff [8] considered the interpolation problem where information is prescribed for higher derivatives which are not consecutive. In this case, unlike Hermite interpolation, a unique solution does not always exist. The $(0,m_1,\ldots,m_{s-1})$ Birkhoff (or lacunary) interpolation problem consists of finding a polynomial Q_{sn} such that

$$Q_{sn}^{(j)}(x_k) = y_k^{(j)}, \quad k=1,\ldots,n, \quad j=0,m_1,\ldots,m_{s-1},$$

where the $y_k^{(j)}$ are given data and $0,m_1,\ldots,m_{s-1}$ are not all consecutive integers. We see that Q_n must satisfy sn conditions, so that $Q_n \in \Pi_{sn-1}$. In general, it is very difficult to find an explicit representation of these polynomials. In fact, it is usually difficult to even determine when there exists a unique solution to this problem.

Let $P_n(x)$ denote the n^{th} Legendre polynomial normalized by $P_n(1) = 1$, and define $\pi_n(x) = (1-x^2)P'_{n-1}(x)$. In 1955, J. Surányi and P. Turán [39] began studying the case of $(0,2)$ interpolation on the zeros of $\pi_n(x)$. They showed that

for n even, the (0,2) interpolation problem has a unique solution. Later J. Balázs and P. Turán [5] provided explicit forms for the fundamental polynomials. They also proved convergence results and Markov-type inequalities with these polynomials. The condition of convergence in this case was later improved by G. Freud [15] and H. Gonska [16], but P. Vértesi [54] has shown that the process is not uniformly convergent for all continuous functions, as the Lebesgue constant of this type of interpolation is of order exactly $O(n)$.

In 1958, R.B. Saxena and A. Sharma [34],[35] extended the results of Turán to $(0,1,3)$ interpolation, and later Saxena [32] extended them to the $(0,1,2,4)$ case. In 1962, Saxena [33] handled the case of 'modified' $(0,2)$ interpolation on the same nodes. Here, by 'modified', we mean that instead of prescribing second derivative information at the endpoints ± 1, we prescribe first derivative information there. We note that in general, 'modified' cases are more easily handled, and lead to the solutions in the 'pure' cases.

A. K. Varma, R.B. Saxena and A. Saxena [52] studied the case of 'modified' $(0,1,4)$ interpolation (second derivatives instead of fourth derivatives are prescribed at ± 1) on the above zeros, showing the Lebesgue constant to be $O(\log n)$. Thus, they conjectured that this process cannot converge for the whole class of continuous functions on $[-1,1]$. Recently, A. Sharma et al. [38] have shown that the Lebesgue constant for the modified $(0,2,3)$ interpolation (first derivatives instead of third derivatives are prescribed at ± 1) on these zeros is also $O(\log n)$.

In looking for Birkhoff interpolation procedures which converge uniformly for all continuous functions on $[-1,1]$, J. Szabados and A. K. Varma [43] considered higher order $(0,M)$ interpolation. Specifically, they were able to show that the $(0,3)$ Birkhoff-Fejér operator on the above zeros converges uniformly for all continuous

functions on $[-1,1]$. More precisely, they proved that for $f \in C[-1,1]$, the polynomial $R_n(x) \in \Pi_{2n-1}$ satisfying

$$R_n(x_k) = f(x_k), \quad R_n'''(x_k) = 0, \quad k = 1, \ldots, n$$

has the property that

$$\|f(x) - R_n(x)\| = O\left(w_3\left(f, \frac{\log^{\frac{1}{3}} n}{n}\right)\right),$$

where the x_k are the zeros of $\pi_n(x)$ and $\omega_3(f, \delta)$ is the third modulus of smoothness of f. Akhlaghi, Chak and Sharma [2] had already proved the existence and uniqueness and provided explicit forms for the fundamental polynomials for the $(0,3)$ case, as well as the $(0,2,3)$ case [1]. We note that the 'modified' (0,2), (0,3) and (0,2,3) cases on the zeros of $\pi_n(x)$ have been generalized [38] to the 'modified' $(0, \ldots, r-2, r)$, $(0, \ldots, r-3, r)$ and $(0, \ldots, r-3, r-1, r)$ cases, respectively.

Given the positive result in the $(0,3)$ case (and the negative results of the others previously mentioned) we turned to consider the situation where we have a similarly 'balanced' process. In particular, using an alternative representation to that given by Sharma et al. [38], we show that the Lebesgue constant for the 'modified' $(0,1,3,4)$ interpolation (second derivatives instead of fourth derivatives are prescribed at ± 1) on the above zeros is $O(1)$. This enables us to prove the uniform convergence of the 'modified' $(0,1,3,4)$ Birkhoff-Fejér operator for the whole class of continuous functions on $[-1,1]$.

1.4 Markov-type Inequalities

In 1889, A.A. Markov [24] proved that for any polynomial P_n of degree $\leq n$ on the interval $[-1,1]$ we have

(1.4.1)

$$||P_n'|| \leq n^2 ||P_n||,$$

where equality holds only for $P_n(x) = cT_n(x)$, where $T_n(x)$ is the n^{th} Chebyshev polynomial. The Russian chemist Mendeleev [26] had settled the question for the

case $n = 2$ in his studies of the specific gravity of a substance as a function of the percentage of the dissolved substance.

In 1892, A.A. Markov's brother W.A. Markov extended the Markov inequality to all higher derivatives (published in German [25] in 1916), providing an inequality sharp for every n. In 1912, S.N. Bernstein [6] improved Markov's inequality by providing the following pointwise estimate

(1.4.2)

$$|P_n'(x)| \leq \frac{n}{\sqrt{1-x^2}}||P_n||.$$

Note that this provides a much sharper inequality, except near the endpoints ± 1. Both the Markov and Bernstein inequalities play a key role in proving convergence theorems, as we shall see later.

The inequalities of A.A. Markov and S.N. Bernstein can be improved if the class of polynomials is restricted in some way. Let us denote by S_n the set of all polynomials whose degree is n and whose zeros are all real and lie outside $(-1, 1)$, and denote by L_n the set of all polynomials of the form

(1.4.3)

$$P_n(x) = \sum_{k=0}^{n} a_k q_{nk}(x), \quad a_k \geq 0 \quad (k = 0, 1, \dots, n)$$

where $q_{nk}(x) = (1+x)^{n-k}(1-x)^k$. In 1940, P. Erdös [10] proved the following refinement of Markov's inequality.

Theorem 1.4.1 (P. Erdös, 1940) Let $P_n \in S_n$. Then we have

$$||P_n'|| \leq \frac{1}{2}en||P_n||,$$

where the constant $\frac{1}{2}e$ cannot be replaced by a smaller one.

In 1937, E. Hille, G. Szegö and J. D. Tamarkin [18] had extended the Markov inequality in the L^p norm, showing there exists a constant A such that for every algebraic polynomial $P_n(x)$ of degree n we have

$$\left(\int_{-1}^{1} |P_n'(x)|^p dx\right)^{\frac{1}{p}} \leq An^2 \left(\int_{-1}^{1} |P_n(x)|^p dx\right)^{\frac{1}{p}}$$

where $p \geq 1$ and A is independent of n and $P_n(x)$. Further, they noted that the problem of obtaining the best constant in the above problem is extremely difficult.

In 1986, P. Erdös and A. K. Varma [11] settled the above inequality for the Lorentz class L_n of polynomials in the L^2 norm as follows.

<u>Theorem 1.4.2</u> (P. Erdös and A. K. Varma, 1986) Let $P_n \in L_n$, $n \geq 2$. Then we have

(1.4.4)
$$\int_{-1}^{1}(P_n'(x))^2dx \leq \frac{n(n-1)(2n+1)}{4(2n-3)}\int_{-1}^{1}(P_n(x))^2dx$$

with equality if and only if $P_n(x) = c(1 \pm x)^{n-1}(1 \mp x)$.

Also, if $P_n \in L_n$, then we have

(1.4.5)
$$\int_{-1}^{1}(1-x^2)(P_n'(x))^2dx \leq \frac{n(n+1)(2n+3)}{4(2n+1)}\int_{-1}^{1}(1-x^2)(P_n(x))^2dx$$

with equality if and only if $P_n(x) = c(1 \pm x)^n$.

It is known [22] that if $P_n \in S_n$, then $P_n \in L_n$ or $-P_n \in L_n$. Thus, (1.4.4) can be viewed as an extension of Theorem 1.4.1 in the L^2 norm. In 1988, G. V. Milovanović and M.S. Petković [28] extended (1.4.5) with the ultraspherical weight $(1-x^2)^\alpha$, $\alpha \geq 1$ ($\alpha > -1$ if $P_n(\pm 1) = 0$). A new proof is provided, and the requirement that $P_n(\pm 1) = 0$ is removed. Then an extension is provided in a weighted L^4 norm.

Let now H_n denote the set of all polynomials of degree n whose zeros are all real and lie inside $[-1, 1]$. In 1939, P. Turán [45] proved this analogue of Markov's inequality.

<u>Theorem 1.4.3</u> (P. Turán, 1939) Let $P_n \in H_n$. Then we have

$$||P_n'|| > \frac{n^{\frac{1}{2}}}{6}||P_n||.$$

This result was later sharpened by J. Eröd [12]. In 1983, A.K. Varma [51] extended the above in the L^2 norm as follows

Theorem 1.4.4 (A. K. Varma, 1983) Let $P_n \in H_n$ and $n = 2m$. Then we have

$$\int_{-1}^{1} (P_n'(x))^2 dx \geq \left(\frac{n}{2} + \frac{3}{4} + \frac{3}{4(n-1)}\right) \int_{-1}^{1} (P_n(x))^2 dx$$

where equality holds if and only if $P_n(x) = c(1-x^2)^m$.

Moreover, if $n = 2m - 1$, then for $n \geq 3$ we have

$$\int_{-1}^{1} (P_n'(x))^2 dx \geq \left(\frac{n}{2} + \frac{3}{4} + \frac{5}{4(n-2)}\right) \int_{-1}^{1} (P_n(x))^2 dx$$

where holds equality if and only if $P_n(x) = (1 \pm x)^m (1 \mp x)^{m-1}$.

Earlier, A.K. Varma had given asymptotically sharp versions [48],[49] of this result, as well as proved the following [50].

Theorem 1.4.5 (A.K. Varma, 1979) Let $P_n \in H_n$. Then we have $(n = 2m)$

$$\int_{-1}^{1} (1-x^2)(P_n'(x))^2 dx \geq \left(\frac{n}{2} + \frac{1}{4} - \frac{1}{4(n+1)}\right) \int_{-1}^{1} (1-x^2)(P_n(x))^2 dx,$$

where equality holds if and only if $P_n(x) = c(1-x^2)^m$.

We extend this result in the ultraspherical weight $\omega(x) = (1-x^2)^\alpha$, $\alpha > 1$, ($\alpha > -1$ if $P_n(\pm 1) = 0$). By a result of S.P. Zhou [56], for $P_n \in H_n$ and $1 \leq p < \infty$ there exists a constant B, independent of n and P_n, such that

$$\left(\int_{-1}^{1} |P_n'(x)|^p dx\right)^{\frac{1}{p}} \geq B n^{\frac{1}{2}} \left(\int_{-1}^{1} |P_n(x)|^p dx\right)^{\frac{1}{p}}.$$

We provide an asymptotically sharp result for p an even integer, showing in the limit that

$$B^p = \frac{(p-1)(p-3)\cdots 5 \cdot 3}{p^{\frac{p}{2}}}.$$

CHAPTER 2
BIRKHOFF INTERPOLATION : (0,1,3,4) CASE

2.1 Preliminaries

The objective of this chapter is prove that the problem of 'modified' (0,1,3,4) interpolation on the zeros of the polynomial $\pi_n(x)$ has a unique solution, and to provide an explicit representation in this case. By 'modified', we mean that instead of prescribing fourth derivatives at the endpoints ± 1, we prescribe second derivatives there. First we take care of some preliminary items.

Let $P_n(x)$ denote the n^{th} Legendre polynomial normalized by $P_n(1) = 1$. Let

(2.1.1)

$$\pi_n(x) = (1 - x^2)P'_{n-1}(x)$$

and enumerate the zeros of $\pi_n(x)$ by

(2.1.2)

$$-1 = x_n < x_{n-1} < \ldots < x_1 = 1.$$

We list the following known identities. These can be found in the book of G. Sansone [31].

(2.1.3)

$$[(1 - x^2)P'_{n-1}(x)]' + n(n-1)P_{n-1}(x) = 0$$

(2.1.4)

$$xP'_{n-1}(x) - P'_{n-2}(x) = (n-1)P_{n-1}(x)$$

(2.1.5)

$$(1 - x^2)P'_{n-1}(x) = (n-1)[P_{n-2}(x) - xP_{n-1}(x)]$$

(2.1.6)

$$P_n'(x) - P_{n-2}'(x) = (2n-1)P_{n-1}(x)$$

(2.1.7)

$$(1-x^2)\pi_n''(x) + n(n-1)\pi_n(x) = 0$$

(2.1.8)

$$\ell_\nu(x) = \frac{\pi_n(x)}{(x-x_\nu)\pi_n'(x_\nu)}$$

(2.1.9)

$$(x-x_\nu)\ell_\nu^{(k)}(x) + k\ell_\nu^{(k-1)}(x) = \frac{\pi_n^{(k)}(x)}{\pi_n'(x_\nu)}$$

and we make note of the following values

(2.1.10)

$$P_{n-1}'(1) = \frac{n(n-1)}{2} \qquad P_{n-1}''(1) = \frac{(n+1)n(n-1)(n-2)}{8}$$

(2.1.11)

$$\pi_n'(1) = -n(n-1) \quad \pi_n''(1) = \frac{-n^2(n-1)^2}{2} \quad \pi_n'''(1) = \frac{-n^2(n-1)^2(n+1)(n-2)}{8}$$

(2.1.12)

$$\ell_1'(1) = \frac{n(n-1)}{4} \qquad \ell_1''(1) = \frac{(n+1)n(n-1)(n-2)}{24}$$

$$\ell_1'''(1) = \frac{(n+2)(n+1)n(n-1)(n-2)(n-3)}{192}$$

(2.1.13)

$$\ell_\nu'(x_\nu) = 0 \qquad \ell_\nu''(x_\nu) = -\frac{n(n-1)}{3(1-x_\nu^2)}$$

$$\ell_\nu'''(x_\nu) = -\frac{n(n-1)x_\nu}{(1-x_\nu^2)^2}$$

$$\ell_\nu^{(4)}(x_\nu) = \frac{n(n-1)}{5(1-x_\nu^2)^2}(n^2-n+18-\frac{24}{(1-x_\nu^2)}), \quad \nu = 2,\dots,n-1.$$

Observe from (2.1.7) that $\pi_n''(x_\nu) = 0, \quad \nu = 2,\dots,n-1.$

2.2 Existence and Uniqueness

We shall prove the following.

Theorem 2.1 Let a_ν, b_ν, d_ν, $\nu = 1, \ldots, n$; c_1, c_n, and e_ν, $\nu = 2, ..., n-1$ be given real numbers. Then there exists a unique polynomial $Q_n(x)$ of degree $\leq 4n-1$ such that

(2.2.1)

$$Q_n(x_\nu) = a_\nu, \quad Q_n'(x_\nu) = b_\nu, \quad \nu = 1, \ldots, n$$
$$Q_n''(1) = c_1, \quad Q''_n(-1) = c_n,$$
$$Q_n'''(x_\nu) = d_\nu, \quad \nu = 1, \ldots, n$$
$$Q_n^{(4)}(x_\nu) = e_\nu, \quad \nu = 2, \ldots, n-1.$$

From linear algebra, this is equivalent to proving that if $Q_n(x)$ is a polynomial of degree $\leq 4n-1$ satisfying

(2.2.2)

$$Q_n(x_\nu) = Q_n'(x_\nu) = Q_n'''(x_\nu) = 0, \quad \nu = 1, \ldots, n$$
$$Q_n''(\pm 1) = 0,$$
$$Q_n^{(4)}(x_\nu) = 0, \quad \nu = 2, \ldots, n-1,$$

then $Q_n \equiv 0$.

Before we proceed with the proof, we note that for the polynomial $\pi_n^2(x)$ we have

(2.2.3)

$$\pi_n^2(x_\nu) = [\pi_n^2(x)]'_{x_\nu} = 0, \quad [\pi_n^2(x)]''_{x_\nu} = 2\pi_n'(x_\nu)^2, \quad \nu = 1, \ldots, n,$$

$$[\pi_n^2(x)]'''_{x_\nu} = 0, \quad [\pi_n^2(x)]^{(4)}_{x_\nu} = \frac{-8n(n-1)\pi_n'(x_\nu)^2}{1-x_\nu^2}, \quad \nu = 2, \ldots, n-1.$$

Proof of Existence and Uniqueness

Suppose that $Q_n \in \Pi_{4n-1}$ satisfies the conditions in (2.2.2). We show that then $Q_n \equiv 0$. Since $Q_n(x_\nu) = Q_n'(x_\nu) = 0$ for $\nu = 1, \ldots, n$, we write $Q_n(x) = \pi_n^2(x) q_{2n-1}(x)$, where $q_{2n-1} \in \Pi_{2n-1}$. Then we have

$$Q_n''(\pm 1) = \pi_n^2(\pm 1) q_{2n-1}''(\pm 1) + 2[\pi_n^2(x)]'_{\pm 1} q_{2n-1}'(\pm 1) + [\pi_n^2(x)]''_{\pm 1} q_{2n-1}(\pm 1)$$

$$= [\pi_n^2(x)]''_{\pm 1} q_{2n-1}(\pm 1).$$

As $Q_n''(\pm 1) = 0$ and $[\pi_n^2(x)]''_{\pm 1} = 2\pi_n'(\pm 1)^2 \neq 0$, we have $q_{2n-1}(\pm 1) = 0$.
Next,

$$Q_n'''(x_\nu) = \pi_n^2(x_\nu) q_{2n-1}'''(x_\nu) + 3[\pi_n^2(x)]'_{x_\nu} q_{2n-1}''(x_\nu) + 3[\pi_n^2(x)]''_{x_\nu} q_{2n-1}'(x_\nu) + [\pi_n^2(x)]'''_{x_\nu} q_{2n-1}(x_\nu)$$

$$= 3[\pi_n^2(x)]''_{x_\nu} q_{2n-1}'(x_\nu), \quad \nu = 1, \ldots, n,$$

and since $[\pi_n^2(x)]''_{x_\nu} = 2\pi_n'(x_\nu)^2 \neq 0$, we have $q_{2n-1}'(x_\nu) = 0$ for $\nu = 1, \ldots, n$. From the paper of Aklaghi, Chak and Sharma [1], page 63, we have $q_{2n-1} \in \Pi_{2n-2}$, and there exists $s_{n-1} \in \Pi_{n-1}$ such that

(2.2.4)

$$q_{2n-1}(x) = \pi_n(x) s_{n-1}'(x) - \pi_n'(x) s_{n-1}(x)$$

where $s_{n-1}(\pm 1) = 0$. Using (2.1.7), we have

(2.2.5)

$$\pi_n'''(x_\nu) = -\frac{n(n-1)}{1-x_\nu^2} \pi_n'(x_\nu).$$

Thus,

$$q''_{2n-1}(x_\nu) = \pi'_n(x_\nu)s''_{n-1}(x_\nu) - \pi'''_n(x_\nu)s_{n-1}(x_\nu)$$

$$= \frac{\pi'_n(x_\nu)}{1-x_\nu^2}\left((1-x_\nu^2)s''_{n-1}(x_\nu) + n(n-1)s_{n-1}(x_\nu)\right).$$

Now,

$$Q_n^{(4)}(x_\nu) = 6[\pi_n^2(x)]''_{x_\nu} q''_{2n-1}(x_\nu) + [\pi_n^2(x)]^{(4)}_{x_\nu} q_{2n-1}(x_\nu)$$

$$= \frac{12\pi'_n(x_\nu)^3}{1-x_\nu^2}\left((1-x_\nu^2)s''_{n-1}(x_\nu) + n(n-1)s_{n-1}(x_\nu)\right)$$

$$+\frac{8\pi'_n(x_\nu)^3}{1-x_\nu^2} n(n-1)s_{n-1}(x)$$

$$= \frac{4\pi'_n(x_\nu)}{1-x_\nu^2}\left(3(1-x_\nu^2)s''_{n-1}(x_\nu) + 5n(n-1)s_{n-1}(x_\nu)\right) = 0,$$

$$\nu = 2,\ldots,n-1.$$

As $\pi'_n(x_\nu) \neq 0$, we have

(2.2.6)

$$3(1-x_\nu^2)s''_{n-1}(x_\nu) + 5n(n-1)s_{n-1}(x_\nu) = 0, \quad \nu = 2,\ldots,n-1.$$

In fact, since $s_{n-1}(\pm 1) = 0$, (2.2.6) holds for $\nu = 1,\ldots,n$. Then

$$3(1-x^2)s''_{n-1}(x) + 5n(n-1)s_{n-1}(x)$$

is a polynomial of degree at most $n-1$ having n zeros. We conclude

(2.2.7)

$$3(1-x^2)s''_{n-1}(x) + 5n(n-1)s_{n-1}(x) \equiv 0.$$

Since $s_{n-1}(\pm 1) = 0$, we can write

(2.2.8)

$$s_{n-1}(x) = \sum_{\nu=2}^{n-1} z_\nu \pi_\nu(x).$$

Replacing (2.2.8) in (2.2.7) yields

$$\sum_{\nu=2}^{n-1}[5n(n-1)-3\nu(\nu-1)]z_\nu\pi_\nu(x)=0$$

so that $z_\nu = 0$ for $\nu = 2,\dots,n-1$. Thus, $s_{n-1}\equiv 0$ and we deduce $q_{2n-1}\equiv 0$. This gives $Q_n \equiv 0$. Theorem 2.1 follows. □

2.3 Explicit Representation

The polynomial Q_n in the Theorem 2.1 will evidently have the form

$$Q_n(x)=\sum_{\nu=1}^{n}a_\nu A_\nu(x)+\sum_{\nu=1}^{n}b_\nu B_\nu(x)+c_1C_1(x)+c_nC_n(x)+\sum_{\nu=1}^{n}d_\nu D_\nu(x)+\sum_{\nu=2}^{n-1}e_\nu E_\nu(x)$$

where the uniquely determined polynomials $A_\nu(x), B_\nu(x), D_\nu(x)\quad (\nu=1,\dots,n)$, $C_1(x)$, $C_n(x)$, $E_\nu(x)\quad (\nu=2,\dots,n-1)$ of degree $\le 4n-1$ are characterized by the conditions

(2.3.1)

$$A_\nu(x_k)=\delta_{k\nu},\quad A'_\nu(x_k)=A'''_\nu(x_k)=0\quad (k=1,\dots,n)$$

$$A''_\nu(1)=A''_\nu(-1)=0,\quad A^{(4)}_\nu(x_k)=0\quad (k=2,\dots,n-1)$$

(2.3.2)

$$B_\nu(x_k)=0,\quad B'_\nu(x_k)=\delta_{k\nu},\quad B'''_\nu(x_k)=0\quad (k=1,\dots,n)$$

$$B''_\nu(1)=B''_\nu(-1)=0,\quad B^{(4)}_\nu(x_k)=0\quad (k=2,\dots,n-1)$$

(2.3.3)

$$C_1(x_k)=C'_1(x_k)=C'''_1(x_k)=0\quad (k=1,\dots,n)$$

$$C''_1(1)=1,\quad C''_1(-1)=0,\quad C^{(4)}_1(x_k)=0\quad (k=2,\dots,n-1)$$

(2.3.4)

$$C_n(x_k)=C'_n(x_k)=C'''_n(x_k)=0\quad (k=1,\dots,n)$$

$$C_n''(1)=0,\quad C_n''(-1)=1,\quad C_n^{(4)}(x_k)=0 \quad (k=2,\dots,n-1)$$

(2.3.5)

$$D_\nu(x_k)=D_\nu'(x_k)=0,\quad D_\nu'''(x_k)=\delta_{k\nu} \quad (k=1,\dots,n)$$

$$D_\nu''(1)=D_\nu''(-1)=0,\quad D_\nu^{(4)}(x_k)=0 \quad (k=2,\dots,n-1)$$

(2.3.6)

$$E_\nu(x_k)=E_\nu'(x_k)=E_\nu'''(x_k)=0 \quad (k=1,\dots,n)$$

$$E_\nu''(1)=E_\nu''(-1)=0,\quad E_\nu^{(4)}(x_k)=\delta_{k\nu} \quad (k=2,\dots,n-1)$$

Theorem 2.2 The fundamental polynomials of the 'modified' $(0,1,3,4)$ interpolation based on the zeros of the polynomial $\pi_n(x)$ can be explicitly represented in the following manner.

(2.3.7)

$$E_\nu(x)=\frac{\pi_n^2(x)(1-x_\nu^2)}{4n^4(n-1)^4P_{n-1}^5(x_\nu)}\sum_{k=2}^{n-1}\frac{(2k-1)P_{k-1}'(x_\nu)}{k(k-1)\lambda_k}\Big(\pi_n'(x)\pi_k(x)-\pi_n(x)\pi_k'(x)\Big)$$

where $\lambda_k=5n(n-1)-3k(k-1)$, $k=2,\dots,n-1$, $\quad \nu=2,\dots,n-1$

(2.3.8)

$$D_\nu(x)=\frac{\pi_n^3(x)\ell_\nu(x)}{6\pi_n'(x_\nu)^3}-\frac{4}{P_{n-1}^3(x_\nu)}\sum_{k=2}^{n-1}P_{n-1}^3(x_k)\ell_\nu'(x_k)E_k(x) \quad (\nu=1,\dots,n)$$

(2.3.9)

$$C_1(x)=\frac{\pi_n^2(x)\ell_1^2(x)}{2n^2(n-1)^2}+\frac{(1-x)\pi_n^2(x)\ell_1^2(x)}{4n(n-1)}-\frac{3n(n-1)}{2}D_1(x)$$
$$-12\sum_{k=2}^{n-1}P_{n-1}^2(x_k)\ell_1'(x_k)^2\left(1-\frac{n(n-1)}{2}(x_k-1)\right)E_k(x)$$

$$C_n(x)=C_1(-x)$$

(2.3.10)

$$B_1(x) = -\frac{\pi_n(x)\ell_1^3(x)}{n(n-1)} - 2n(n-1)C_1(x) - \frac{n(n-1)(13n^2-13n+1)}{8}D_1(x)$$

$$-24\sum_{k=2}^{n-1} P_{n-1}(x_k)\ell_1'(x_k)^3 E_k(x)$$

$$B_n(x) = -B_1(-x)$$

(2.3.11)

$$B_\nu(x) = \frac{\pi_n(x)\ell_\nu^3(x)}{\pi_n'(x_\nu)} + \frac{4n(n-1)}{1-x_\nu^2}D_\nu(x) - \frac{13n(n-1)x_\nu}{(1-x_\nu^2)^2}E_\nu(x)$$

$$+\frac{24}{P_{n-1}(x_\nu)}\sum_{k=2}^{n-1} P_{n-1}(x_k)\ell_\nu'(x_k)^3 E_k(x)$$

$$(\nu = 2, \ldots, n-1)$$

(2.3.12)

$$A_1(x) = \ell_1^4(x) - n(n-1)B_1(x) - 4(3\ell_1'(1)^2 + \ell_1''(1))C_1(x)$$

$$-4(6\ell_1'(1)^3 + 9\ell_1'(1)\ell_1''(1) + \ell_1'''(1))D_1(x) - 24\sum_{k=2}^{n-1} \ell_1'(x_k)^4 E_k(x)$$

$$A_n(x) = A_1(-x)$$

(2.3.13)

$$A_\nu(x) = \ell_\nu^4(x) + \frac{4n(n-1)x_\nu}{(1-x_\nu^2)^2}D_\nu(x) + \frac{24n(n-1)}{5(1-x_\nu^2)^2}\left(n^2 - n + 3 - \frac{4}{1-x_\nu^2}\right)E_\nu(x)$$

$$-24\sum_{k=2}^{n-1} \ell_\nu'(x_k)^4 E_k(x) \quad (v = 2, \ldots, n-1).$$

Proof of Theorem 2.2

We first provide the proof of the representation of the last fundamental polynomials. Following the proof of Theorem 2.1, we write $E_\nu(x) = \pi_n^2(x)q_{2n-1}(x)$ where $q_{2n-1}(x) = \pi_n(x)s'_{n-1}(x) - \pi'_n(x)s_{n-1}(x) \in \Pi_{2n-2}$. Then, instead of (2.2.6) we get

$$\frac{4\pi'_n(x_k)^3}{1-x_k^2}\left(3(1-x_k^2)s''_{n-1}(x_k) + 5n(n-1)s_{n-1}(x_k)\right) = \delta_{k\nu}, \quad k = 2,\ldots,n-1,$$

or equivalently,

$$3(1-x_k^2)s''_{n-1}(x_k) + 5n(n-1)s_{n-1}(x_k) = \frac{1-x_\nu^2}{4\pi'_n(x_\nu)^3}\ell_\nu(x_k), \quad k = 1,\ldots,n.$$

Hence,

$$3(1-x^2)s''_{n-1}(x) + 5n(n-1)s_{n-1}(x) \equiv \frac{1-x_\nu^2}{4\pi'_n(x_\nu)^3}\ell_\nu(x).$$

Now, using the identity

$$\ell_\nu(x) = \frac{1}{n(n-1)P_{n-1}^2(x_\nu)}\sum_{k=2}^{n-1}\frac{2k-1}{k(k-1)}P'_{k-1}(x_\nu)\pi_k(x), \quad \nu = 2,\ldots,n-1,$$

which can be found in the (0,2,3) paper of Akhlaghi, Chak and Sharma [1], page 58, and using the representation (2.2.8) we obtain

$$z_k = \frac{n(n-1)(1-x_\nu^2)(2k-1)P'_{k-1}(x_\nu)}{4k(k-1)\lambda_k\pi'_n(x_\nu)^5}$$

so that

$$s_{n-1}(x) = \frac{n(n-1)(1-x_\nu^2)}{4\pi'_n(x_\nu)^5}\sum_{\nu=2}^{n-1}\frac{(2k-1)P'_{k-1}(x_\nu)}{k(k-1)\lambda_k}\pi_k(x),$$

and we have (2.3.7).

To verify (2.3.8) we observe (2.3.5), (2.3.6) and that for the polynomial $\pi_n^3(x)\ell_\nu(x)$ we have

$$\pi_n^3(x_k)\ell_\nu(x_k) = [\pi_n^3(x)\ell_\nu(x)]'_{x_k} = [\pi_n^3(x)\ell_\nu(x)]''_{x_k} = 0 \quad k = 1,\ldots,n$$

$$[\pi_n^3(x)\ell_\nu(x)]'''_{x_k} = 6\pi_n'(x_\nu)^3\delta_{k\nu} \quad k = 1,\ldots,n$$

$$[\pi_n^3(x)\ell_\nu(x)]^{(4)}_{x_k} = \frac{4\pi_n'(x_k)^3\ell_\nu'(x_k)}{\pi_n'(x_\nu)^3} \quad k = 2,\ldots,n-1.$$

To verify (2.3.9), we observe (2.3.3), (2.3.5), (2.3.6) and that for the polynomial

$$\frac{\pi_n^2(x)\ell_1^2(x)}{2n^2(n-1)^2} + \frac{(1-x)\pi_n^2(x)\ell_1^2(x)}{4n(n-1)}$$

we have

$$\frac{\pi_n^2(x_k)\ell_1^2(x_k)}{2n^2(n-1)^2} + \frac{(1-x_k)\pi_n^2(x_k)\ell_1^2(x_k)}{4n(n-1)} = 0,$$

$$\left[\frac{\pi_n^2(x)\ell_1^2(x)}{2n^2(n-1)^2} + \frac{(1-x)\pi_n^2(x)\ell_1^2(x)}{4n(n-1)}\right]'_{x_k} = 0, \quad k = 1,\ldots,n,$$

$$\left[\frac{\pi_n^2(x)\ell_1^2(x)}{2n^2(n-1)^2} + \frac{(1-x)\pi_n^2(x)\ell_1^2(x)}{4n(n-1)}\right]''_{1} = 1,$$

$$\left[\frac{\pi_n^2(x)\ell_1^2(x)}{2n^2(n-1)^2} + \frac{(1-x)\pi_n^2(x)\ell_1^2(x)}{4n(n-1)}\right]''_{-1} = 0$$

$$\left[\frac{\pi_n^2(x)\ell_1^2(x)}{2n^2(n-1)^2} + \frac{(1-x)\pi_n^2(x)\ell_1^2(x)}{4n(n-1)}\right]'''_{x_k} = \frac{3n(n-1)}{2}\delta_{k1}, \quad k = 1,\ldots,n,$$

$$\left[\frac{\pi_n^2(x)\ell_1^2(x)}{2n^2(n-1)^2} + \frac{(1-x)\pi_n^2(x)\ell_1^2(x)}{4n(n-1)}\right]^{(4)}_{x_k} = \frac{12\pi_n'(x_k)^2\ell'(x_k)^2(1-\frac{n(n-1)}{2}(x_k-1))}{n^2(n-1)^2},$$

$$k = 2,\ldots,n-1.$$

To verify (2.3.10) we observe (2.3.2)–(2.3.6) and that for the polynomial $\pi_n(x)\ell_1^3(x)$ we have

$$\pi_n(x_k)\ell_1^3(x_k) = 0, \quad [\pi_n(x)\ell_1^3(x)]'_{x_k} = \pi'_n(1)\delta_{k1} = -n(n-1)\delta_{k1},$$

$$[\pi_n(x)\ell_1^3(x)]''_{x_k} = -2n^2(n-1)^2\delta_{k1},$$

$$[\pi_n(x)\ell_1^3(x)]'''_{x_k} = -\frac{n^2(n-1)^2(13n^2-13n+1)}{8}\delta_{k1}, \quad k = 1, \dots, n,$$

$$[\pi_n(x)\ell_1^3(x)]^{(4)}_{x_k} = 24\pi'_n(x_k)\ell'_1(x_k)^3, \quad k = 2, \dots, n-1.$$

To verify (2.3.11) we observe (2.3.2), (2.3.5), (2.3.6) and that for the polynomial $\pi_n(x)\ell_\nu^3(x)$ we have

$$\pi_n(x_k)\ell_\nu^3(x_k) = [\pi_n(x)\ell_\nu^3(x)]'_{x_k} = 0, \quad k = 1, \dots, n$$

$$[\pi_n(x)\ell_\nu^3(x)]''_{x_k} = \pi'_n(x_\nu)\delta_{k\nu}, \quad [\pi_n(x)\ell_\nu^3(x)]'''_{x_k} = -\frac{4n(n-1)\pi'_n(x_\nu)}{1-x_\nu^2}\delta_{k\nu} \quad k = 1, \dots, n$$

$$[\pi_n(x)\ell_\nu^3(x)]^{(4)}_{x_k} = 24\pi'_n(x_k)\ell'_\nu(x_k)^3, \quad k = 2, \dots, n-1, \quad k \neq \nu,$$

$$[\pi_n(x)\ell_\nu^3(x)]^{(4)}_{x_\nu} = -\frac{13n(n-1)x_\nu\pi'_n(x_\nu)}{(1-x_\nu^2)^2}, \quad \nu = 2, \dots, n-1.$$

To verify (2.3.12), we observe (2.3.1)–(2.3.6) and that for the polynomial $\ell_1^4(x)$ we have

$$\ell_1^4(x_k) = \delta_{k1}, \quad [\ell_1^4(x)]'_{x_k} = 4\ell_1'(1)\delta_{k1} = n(n-1)\delta_{k1}, \quad k = 1, \ldots, n,$$

$$[\ell_1^4(x)]''_{x_k} = (12\ell_1'(1)^2 + 4\ell_1''(1))\delta_{k1} = \frac{n(n-1)(11n^2 - 11n - 4)}{12}\delta_{k1}, \quad k = 1, \ldots, n,$$

$$[\ell_1^4(x)]'''_{x_k} = (24\ell_1'(1)^3 + 36\ell_1'(1)\ell_1''(1) + 4\ell_1'''(1))\delta_{k1}, \quad k = 1, \ldots, n,$$

$$[\ell_1^4(x)]^{(4)}_{x_k} = 24\ell_1'(x_k)^4, \quad k = 2, \ldots, n-1.$$

Finally, to verify (2.3.13) we observe (2.3.1), (2.3.5), (2.3.6) and that

$$\ell_\nu^4(x_k) = \delta_{k\nu}, \quad [\ell_\nu^4(x)]'_{x_k} = 0, \quad k = 1, \ldots, n,$$

$$[\ell_\nu^4(x)]''_{\pm 1} = 0, \quad [\ell_\nu^4(x)]'''_{x_k} = -\frac{4n(n-1)x_\nu}{(1 - x_\nu^2)^2}\delta_{k\nu}, \quad k = 1, \ldots, n,$$

$$[\ell_\nu^4(x)]^{(4)}_{x_k} = 24\ell_\nu'(x_k)^4, \quad k = 2, \ldots, n-1, \quad k \neq \nu,$$

$$[\ell_\nu^4(x)]^{(4)}_{x_\nu} = 36\ell_\nu''(x_\nu)^2 + 4\ell_\nu^{(4)}(x_\nu) = \frac{24n(n-1)}{5(1 - x_\nu^2)^2}(n^2 - n + 3 - \frac{4}{1 - x_\nu^2}) \quad \nu = 2, \ldots, n-1.$$

This completes the proof of Theorem 2.2. $\square$

CHAPTER 3
CONVERGENCE RESULTS FOR A BIRKHOFF-FEJÉR OPERATOR

3.1 Preliminaries and Convergence Theorem

Let f be a real-valued function on the interval $[-1,1]$, and define the 'modified' $(0,1,3,4)$ operator

$$R_n(f,x) = \sum_{\nu=1}^{n} f(x_\nu)A_\nu(x),$$

where the fundamental polynomials of the first kind $A_\nu(x)$ $(\nu = 1,\ldots,n)$ are given by (2.3.12)–(2.3.13).

The main goal of this chapter is to prove the following.

Theorem 3.1 Let f be a continuous function on the interval $[-1,1]$. Then

$$\|f(x) - R_n(f,x)\| = O\left(w_1\left(f, \frac{\log n}{n}\right)\right),$$

where $\omega_1(f,\delta)$ is the first modulus of smoothness of f.

Observe that the rate of convergence is the same as that of the classical Hermite-Fejér operator, but it is not as good as that of the $(0,3)$ Birkhoff-Fejér operator on the zeros of $\pi_n(x)$.

We shall employ the following notations

(3.1.1)

$$x = \cos t,\ \ x_k = \cos t_k \quad (k = 1,\ldots,n)$$

(3.1.2)

$$0 = t_n < t_{n-1} < \ldots < t_1 = \pi.$$

We shall use the following known estimates.

(3.1.3)
$$|P_{n-1}(x)| = O\left(\frac{1}{(1-x^2)^{\frac{1}{4}} n^{\frac{1}{2}}}\right), \qquad -1 \le x \le 1,$$

(3.1.4)
$$|P'_{n-1}(x)| = O\left(\frac{n^{\frac{1}{2}}}{(1-x^2)^{\frac{3}{4}}}\right), \qquad -1 \le x \le 1,$$

(3.1.5)
$$|\pi_n(x)| = O\left((1-x^2)^{\frac{1}{4}} n^{\frac{1}{2}}\right), \qquad -1 \le x \le 1.$$

These can be found in Szegö's book [44].

(3.1.6)
$$|P_n(x) + P_{n+1}(x)| = O\left(\frac{\sin^{\frac{1}{2}} t}{n^{\frac{1}{2}}}\right), \qquad -1 \le x \le \frac{1}{2}.$$

This is a result from the paper of Szabados and Varma [43], page 734. There exist absolute positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that

(3.1.7)
$$\alpha_1 \frac{\nu^2}{(n-1)^2} < 1 - x_\nu^2 < \alpha_2 \frac{\nu^2}{(n-1)^2}, \quad \nu = 2, \ldots, \left[\frac{n-1}{2}\right],$$

(3.1.8)
$$\alpha_1 \frac{(n-\nu)^2}{(n-1)^2} < 1 - x_\nu^2 < \alpha_2 \frac{(n-\nu)^2}{(n-1)^2}, \quad \nu = \left[\frac{n-1}{2}\right] + 1, \ldots, n-1,$$

(3.1.9)
$$P_{n-1}^2(x_\nu) > \frac{\alpha_3}{n \sin t_\nu}, \quad \nu = 2, \ldots, n-1,$$

(3.1.10)
$$|t_\nu - t_{\nu+1}| > \frac{\alpha_4}{n},$$

(3.1.11)
$$\sin t \le \sin t + \sin t_\nu \le 2 \sin \frac{t + t_\nu}{2},$$
$$\sin t_\nu \le \sin t + \sin t_\nu \le 2 \sin \frac{t + t_\nu}{2}.$$

We shall refer several times to using an Abel transformation with factors b_k, by which we mean that we are using the summation by parts formula

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q-1} A_k(b_k - b_{k+1}) + A_q b_q + A_{p-1} b_p,$$

$$\text{where } A_k = \sum_{s=0}^{k} a_s.$$

3.2 Estimate of the Fundamental Polynomials of the Fourth Kind

Lemma 3.2.1 For the fundamental polynomials $E_\nu(x)$ $(\nu = 2, \ldots, n-1)$ we have the following estimate

$$|E_\nu(x)| = \begin{cases} O\left(\frac{\sin^3 t_\nu}{n^6 \sin \frac{t-t_\nu}{2}} + \frac{\sin^4 t_\nu}{n^6 \sin^2 \frac{t-t_\nu}{2}}\right), & \text{if } |t - t_\nu| > \frac{c}{n}, \\ O\left(\frac{\sin^4 t_\nu}{n^4}\right), & \text{if } |t - t_\nu| \le \frac{c}{n}, \end{cases}$$

where $c > 0$ is an absolute constant.

Proof

We shall first prove the case when $|t - t_\nu| > \frac{c}{n}$. Using (2.3.7), (2.1.1) and (2.1.3) we have

$$E_\nu(x) = \frac{\pi_n^2(x)(1-x_\nu^2)}{4n^4(n-1)^4 P_{n-1}^5(x_\nu)} \Big[(1-x^2)\pi_n'(x) \sum_{k=2}^{n-1} \frac{2k-1}{k(k-1)\lambda_k} P_{k-1}'(x_\nu) P_{k-1}'(x)$$

$$+\pi_n(x) \sum_{k=2}^{n-1} \frac{2k-1}{\lambda_k} P_{k-1}'(x_\nu) P_{k-1}(x)\Big]$$

$$\equiv S_1 + S_2.$$

Following the same argument given in the paper of Szabados and Varma [43], pages 736-738, with $\lambda_k = 5n(n-1) - 3k(k-1)$ we obtain

$$\sum_{k=2}^{n-1} \frac{(2k-1)}{k(k-1)\lambda_k} P_{k-1}'(x_\nu) P_{k-1}'(x) = -\frac{P_{n-1}'(x) P_{n-1}(x_\nu)}{\lambda_{n-1}(x - x_\nu)}$$

$$+O\left(\frac{\sin\frac{|t-t_\nu|}{2}+\sin t_\nu}{n^3\sin^{\frac{3}{2}}t\ \sin^{\frac{3}{2}}t_\nu\sin^2\frac{t-t_\nu}{2}\sin\frac{t+t_\nu}{2}}\right).$$

Thus,

$$S_1=-\frac{\pi_n^2(x)(1-x_\nu^2)(1-x^2)\pi_n'(x)P_{n-1}'(x)P_{n-1}'(x_\nu)}{4n^4(n-1)^4P_{n-1}^5(x_\nu)\lambda_{n-1}(x-x_\nu)}$$
$$+\frac{\pi_n^2(x)(1-x_\nu^2)(1-x^2)\pi_n'(x)}{4n^4(n-1)^4P_{n-1}^5(x_\nu)}\cdot O\left(\frac{\sin\frac{|t-t_\nu|}{2}+\sin t_\nu}{n^3\sin^{\frac{3}{2}}t\sin^{\frac{3}{2}}t_\nu\sin^2\frac{t-t_\nu}{2}\sin\frac{t+t_\nu}{2}}\right).$$

The first term in S_1 will later cancel with a term occurring in S_2. Using (2.1.3), (3.1.5), (3.1.9) and (3.1.11) we obtain the estimate

$$O\left(\frac{\sin^3 t_\nu}{n^6\sin\frac{|t-t_\nu|}{2}}+\frac{\sin^4 t_\nu}{n^6\sin^2\frac{t-t_\nu}{2}}\right)$$

for the remaining term in S_1.

We turn now to estimate S_2. Using an Abel transformation with the factors $\frac{1}{\lambda_k}$ yields

(3.2.1)

$$S_2=\frac{\pi_n^3(x)(1-x_\nu^2)}{4n^4(n-1)^4P_{n-1}^5(x_\nu)}\sum_{k=2}^{n-1}\frac{2k-1}{\lambda_k}P_{k-1}'(x_\nu)P_{k-1}(x)$$

$$=\frac{\pi_n^3(x)(1-x_\nu^2)}{4n^4(n-1)^4P_{n-1}^5(x_\nu)}\Bigg[\frac{1}{\lambda_{n-1}}\sum_{k=1}^{n-2}(2k+1)P_k(x)P_k'(x_\nu)$$
$$-6\sum_{k=2}^{n-2}\left(\frac{k}{\lambda_{k+1}\lambda_k}\sum_{s=1}^{k=1}(2s+1)P_s(x)P_s'(x_\nu)\right)\Bigg].$$

Next, we differentiate both sides of the Christoffel–Darboux formula

(3.2.2)

$$\sum_{r=0}^{n}(2r+1)P_r(x)P_r(y)=(n+1)\frac{P_n(x)P_{n+1}(y)-P_{n+1}(x)P_n(y)}{y-x}$$

with respect to y and set x_ν equal to y to obtain

(3.2.3)

$$\sum_{r=0}^{n}(2r+1)P_r(x)P_r'(x_\nu) = (n+1)\Big[\frac{P_n(x)P_{n+1}'(x_\nu) - P_{n+1}(x)P_n'(x_\nu)}{x_\nu - x} - \frac{P_n(x)P_{n+1}(x_\nu) - P_{n+1}(x)P_n(x_\nu)}{(x_\nu - x)^2}\Big],$$

and applying this result to (3.2.1) yields

$$S_2 = \frac{n-1}{\lambda_{n-1}}\Big[\frac{P_{n-2}(x)P_{n-1}'(x_\nu) - P_{n-1}(x)P_{n-2}'(x_\nu)}{x_\nu - x} - \frac{P_{n-2}(x)P_{n-1}(x_\nu) - P_{n-1}(x)P_{n-2}(x_\nu)}{(x_\nu - x)^2}\Big]$$

$$-6\sum_{k=2}^{n-2}\frac{k^2}{\lambda_k\lambda_{k+1}}\left(\frac{P_{k-1}(x)P_k'(x_\nu) - P_k(x)P_{k-1}'(x_\nu)}{x_\nu - x} - \frac{P_{k-1}(x)P_k(x_\nu) - P_k(x)P_{k-1}(x_\nu)}{(x_\nu - x)^2}\right)$$

$$= \frac{(n-1)P_{n-1}(x)P_{n-2}'(x_\nu)}{\lambda_{n-1}(x - x_\nu)} - \frac{(n-1)P_{n-2}(x)P_{n-1}(x_\nu)}{\lambda_{n-1}(x - x_\nu)^2} + \frac{(n-1)P_{n-1}(x)P_{n-2}(x_\nu)}{\lambda_{n-1}(x - x_\nu)^2}$$

$$+\frac{6}{x - x_\nu}\sum_{k=2}^{n-2}\frac{k^2}{\lambda_k\lambda_{k+1}}\Big(P_{k-1}(x)P_k'(x_\nu) - P_k(x)P_{k-1}'(x_\nu)\Big)$$

$$+\frac{6}{(x - x_\nu)^2}\sum_{k=2}^{n-2}\frac{k^2}{\lambda_k\lambda_{k+1}}\Big(P_{k-1}(x)P_k(x_\nu) - P_k(x)P_{k-1}(x_\nu)\Big).$$

From (2.1.4) and (2.1.5) we have

(3.2.4)

$$P_{n-2}'(x_\nu) = -(n-1)P_{n-1}(x_\nu)$$

(3.2.5)

$$P_{n-2}(x_\nu) = x_\nu P_{n-1}(x_\nu)$$

(3.2.6)

$$P_{n-2}(x) = \frac{\pi_n(x)}{n-1} + xP_{n-1}(x).$$

Using (3.2.4)–(3.2.6) we combine the first 3 terms in S_2 to obtain

$$\frac{(n-1)P_{n-1}(x)P'_{n-2}(x_\nu)}{\lambda_{n-1}(x-x_\nu)} - \frac{(n-1)P_{n-2}(x)P_{n-1}(x_\nu)}{\lambda_{n-1}(x-x_\nu)^2} + \frac{(n-1)P_{n-1}(x)P_{n-2}(x_\nu)}{\lambda_{n-1}(x-x_\nu)^2}$$

$$= -\frac{(n-1)^2P_{n-1}(x)P_{n-1}(x_\nu)}{\lambda_{n-1}(x-x_\nu)} - \frac{(n-1)P_{n-1}(x_\nu)}{\lambda_{n-1}(x-x_\nu)^2}\left(\frac{\pi_n(x)}{n-1} + xP_{n-1}(x)\right)$$
$$+ \frac{(n-1)x_\nu P_{n-1}(x)P_{n-1}(x_\nu)}{\lambda_{n-1}(x-x_\nu)^2}$$

$$= -\frac{(n-1)P_{n-1}(x)P_{n-1}(x_\nu)}{\lambda_{n-1}(x-x_\nu)} - \frac{P_{n-1}(x_\nu)\pi_n(x)}{\lambda_{n-1}(x-x_\nu)^2},$$

and multiplying through by $\frac{\pi_n^3(x)(1-x_\nu^2)}{4n^4(n-1)^4P_{n-1}^5(x_\nu)}$ yields

$$-\frac{n(n-1)P_{n-1}(x)P_{n-1}(x_\nu)\pi_n^3(x)(1-x_\nu^2)}{4\lambda_{n-1}(x-x_\nu)n^4(n-1)^4P_{n-1}^5(x_\nu)} - \frac{\pi_n^4(x)(1-x_\nu^2)}{4\lambda_{n-1}(x-x_\nu)^2n^4(n-1)^4P_{n-1}^4(x_\nu)},$$

where the first term above cancels with the aforementioned term in S_1, and we estimate the second term above using (3.1.9), (3.1.5) and (3.1.11) to obtain

$$O\left(\frac{\sin^4 t_\nu}{n^6\sin^2\frac{t-t_\nu}{2}}\right).$$

To complete the first part of the proof, we need only estimate the terms

(3.2.7)

$$T_1 = \frac{3\pi_n^3(x)(1-x_\nu^2)}{2(x-x_\nu)n^4(n-1)^4P_{n-1}^5(x_\nu)}\sum_{k=2}^{n-2}\frac{k^2}{\lambda_k\lambda_{k+1}}\left(P_{k-1}(x)P'_k(x_\nu) - P_k(x)P'_{k-1}(x_\nu)\right)$$

and

(3.2.8)

$$T_2 = \frac{3\pi_n^3(x)(1-x_\nu^2)}{2(x-x_\nu)^2 n^4 (n-1)^4 P_{n-1}^5(x_\nu)} \sum_{k=2}^{n-2} \frac{k^2}{\lambda_k \lambda_{k+1}} \Big(P_{k-1}(x)P_k(x_\nu) - P_k(x)P_{k-1}(x_\nu) \Big)$$

As $P_k(-x) = (-1)^k P_k(x)$ and $x_{n-\nu+1} = -x_\nu$, $\nu = 1, \ldots, n$, we may assume that $-1 < x \leq 0$.

We show first that $|T_2| = O\left(\frac{\sin^4 t_\nu}{n^6 \sin^2 \frac{t-t_\nu}{2}}\right)$. We break this part of the proof into two cases. We begin with the case $-1 < x_\nu \leq \frac{1}{2}$. Notice

(3.2.9)

$$|P_{k-1}(x)P_k(x_\nu) - P_k(x)P_{k-1}(x_\nu)| = |P_k(x_\nu)[P_{k-1}(x)+P_k(x)] - P_k(x)[P_{k-1}(x_\nu)+P_k(x_\nu)]|$$

$$\leq |P_k(x_\nu)||P_{k-1}(x) + P_k(x)| + |P_k(x)||P_{k-1}(x_\nu) + P_k(x_\nu)|,$$

so that applying (3.1.3) and (3.1.6) yields

(3.2.10)

$$|P_{k-1}(x)P_k(x_\nu) - P_k(x)P_{k-1}(x_\nu)| \leq O\left(\frac{\sin^{\frac{1}{2}} t}{k \sin^{\frac{1}{2}} t_\nu} + \frac{\sin^{\frac{1}{2}} t_\nu}{k \sin^{\frac{1}{2}} t} \right).$$

Thus,

$$\left| \sum_{k=2}^{n-2} \frac{k^2}{\lambda_k \lambda_{k+1}} \Big(P_{k-1}(x)P_k(x_\nu) - P_k(x)P_{k-1}(x_\nu) \Big) \right|$$

$$= O\left[\left(\frac{\sin^{\frac{1}{2}} t}{n^4 \sin^{\frac{1}{2}} t_\nu} + \frac{\sin^{\frac{1}{2}} t_\nu}{n^4 \sin^{\frac{1}{2}} t} \right) \sum_{k=2}^{n-2} k \right] = O\left(\frac{\sin^{\frac{1}{2}} t}{n^2 \sin^{\frac{1}{2}} t_\nu} + \frac{\sin^{\frac{1}{2}} t_\nu}{n^2 \sin^{\frac{1}{2}} t} \right).$$

Since

$$O\left(\frac{3\pi_n^3(x)(1-x_\nu^2)}{2n^4(n-1)^4P_{n-1}^5(x_\nu)(x-x_\nu)^2}\right)=O\left(\frac{\sin^{\frac{3}{2}}t\ \sin^{\frac{9}{2}}t_\nu}{n^4\sin^2\frac{t-t_\nu}{2}\sin^2\frac{t+t_\nu}{2}}\right)$$

we get by using (3.1.11)

$$|T_2|=O\left(\frac{\sin^2 t\ \sin^4 t_\nu}{n^6\sin^2\frac{t-t_\nu}{2}\sin^2\frac{t+t_\nu}{2}}+\frac{\sin t\ \sin^5 t_\nu}{n^6\sin^2\frac{t-t_\nu}{2}\sin^2\frac{t+t_\nu}{2}}\right)=O\left(\frac{\sin^4 t_\nu}{n^6\sin^2\frac{t-t_\nu}{2}}\right).$$

Now we consider the case when $\frac{1}{2}\leq x_\nu<1$. Since $-1<x\leq 0$, we have $|x-x_\nu|\geq\frac{1}{2}$. Applying this, (3.2.9), (3.1.5), (3.1.3), (3.1.9) and (3.1.11) to (3.2.8) yields the estimate

$$|T_2|=O\left(\frac{\sin^4 t_\nu}{n^6\sin\frac{|t-t_\nu|}{2}}\right)\leq O\left(\frac{\sin^4 t_\nu}{n^6\sin^2\frac{t-t_\nu}{2}}\right)$$

in this case.

Lastly, we show

$$|T_1|=O\left(\frac{\sin^4 t_\nu}{n^6\sin^2\frac{t-t_\nu}{2}}\right).$$

Using (2.1.6), we observe that

(3.2.11)

$$\hat{S}=\sum_{k=2}^{n-2}\frac{k^2}{\lambda_k\lambda_{k+1}}\left(P_{k-1}(x)P_k'(x_\nu)-P_k(x)P_{k-1}'(x_\nu)\right)$$

$$=\sum_{k=2}^{n-2}\frac{k^2}{\lambda_k\lambda_{k+1}}\left(P_{k-1}(x)P_{k-2}'(x_\nu)+(2k-1)P_{k-1}(x)P_{k-1}(x_\nu)-P_k(x)P_{k-1}'(x_\nu)\right)$$

$$=\sum_{k=2}^{n-2}\frac{k^2P_{k-1}(x)P_{k-2}'(x_\nu)}{\lambda_k\lambda_{k+1}}+\sum_{k=2}^{n-2}\frac{k^2}{\lambda_k\lambda_{k+1}}(2k-1)P_{k-1}(x)P_{k-1}(x_\nu)-\sum_{k=2}^{n-2}\frac{k^2P_k(x)P_{k-1}'(x_\nu)}{\lambda_k\lambda_{k+1}}.$$

Now, combining the first and third sums above, and applying an Abel transformation

with the factors $\frac{(k+1)^2}{\lambda_{k+1}\lambda_{k+2}}$ to the second sum (after reindexing), we have

(3.2.12)

$$\hat{S} = \Big[-\frac{(n-2)^2 P_{n-2}(x)P'_{n-3}(x_\nu)}{\lambda_{n-2}\lambda_{n-1}} + \sum_{k=3}^{n-2} \frac{P_{k-1}(x)P'_{k-2}(x_\nu)}{\lambda_k}\Big(\frac{k^2}{\lambda_{k+1}} - \frac{(k-1)^2}{\lambda_{k-1}}\Big)\Big]$$

$$+\Big[\frac{(n-2)^2}{\lambda_{n-1}\lambda_{n-2}} \sum_{k=1}^{n-3}(2k+1)P_k(x)P_k(x_\nu)$$

$$+\sum_{k=2}^{n-4}\Big(\frac{(k+1)^2}{\lambda_{k+1}\lambda_{k+2}} - \frac{(k+2)^2}{\lambda_{k+2}\lambda_{k+3}}\Big)\sum_{s=1}^{k-1}(2s+1)P_s(x)P_s(x_\nu)\Big].$$

We then apply the Christoffel–Darboux formula (3.2.2) and that

$$\frac{k^2}{\lambda_{k+1}} - \frac{(k-1)^2}{\lambda_{k-1}} = \frac{(2k-1)(5n(n-1)+3k(k-1))}{\lambda_{k+1}\lambda_{k-1}} = \frac{(2k-1)\lambda_k}{\lambda_{k+1}\lambda_{k-1}}$$

to obtain

(3.2.13)

$$\hat{S} = \Big[-\frac{(n-2)^2 P_{n-2}(x)P'_{n-3}(x_\nu)}{\lambda_{n-2}\lambda_{n-1}} + \sum_{k=3}^{n-2}\frac{(2k-1)\lambda_k}{\lambda_{k-1}\lambda_k\lambda_{k+1}} P_{k-1}(x)P'_{k-2}(x_\nu)\Big]$$

$$+\Big[\frac{(n-2)^3}{\lambda_{n-2}\lambda_{n-1}}\Big(\frac{P_{n-3}(x)P_{n-2}(x_\nu) - P_{n-2}(x)P_{n-3}(x_\nu)}{x_\nu - x}\Big)$$

$$-\sum_{k=2}^{n-4}\frac{(2k+3)\lambda_{k+2}}{\lambda_{k+1}\lambda_{k+2}\lambda_{k+3}}\Big(k\frac{P_{k-1}(x)P_k(x_\nu) - P_k(x)P_{k-1}(x_\nu)}{x_\nu - x}\Big)\Big]$$

$$\equiv U_1 + U_2.$$

Using (3.1.3) and (3.1.4), estimating term–by–term we obtain

(3.2.14)

$$|U_1| = O\left(\frac{1}{n^2 \sin^{\frac{1}{2}} t \sin^{\frac{3}{2}} t_\nu}\right).$$

We break the estimate of U_2 into 2 cases. First suppose $-1 < x_\nu \leq \frac{1}{2}$. Applying (3.2.10) to U_2, and estimating term–by–term we get

(3.2.15)
$$|U_2| = O\left(\frac{\sin^{\frac{1}{2}} t_\nu}{n^2|x-x_\nu|\sin^{\frac{1}{2}} t} + \frac{\sin^{\frac{1}{2}} t}{n^2|x-x_\nu|\sin^{\frac{1}{2}} t_\nu}\right).$$

Now, suppose $\frac{1}{2} \leq x_\nu < 1$ (from before we need only consider $-1 < x \leq 0$). Then $\frac{1}{|x-x_\nu|} \leq 2$, and we have on using (3.2.9) and (3.1.3)

(3.2.16)
$$|U_2| = \left(\frac{1}{n^2\sin^{\frac{1}{2}} t\sin^{\frac{1}{2}} t_\nu}\right) + O\left(n^{-4}\sum_{k=2}^{n-4}(2k+3)k\Big(P_{k-1}(x)P_k(x_\nu) - P_k(x)P_{k-1}(x_\nu)\Big)\right)$$

$$= O\left(\frac{1}{n^2\sin^{\frac{1}{2}} t\sin^{\frac{1}{2}} t_\nu} + \frac{\sum_{k=2}^{n-4}(2k+3)}{n^4\sin^{\frac{1}{2}} t\sin^{\frac{1}{2}} t_\nu}\right)$$

$$= O\left(\frac{1}{n^2\sin^{\frac{1}{2}} t\sin^{\frac{1}{2}} t_\nu}\right).$$

Thus, from (3.2.7), (3.2.11)–(3.2.16), and then (3.1.9), (3.1.5) and (3.1.11) we obtain

$$|T_1| = \left|\frac{3\pi_n^3(x)(1-x_\nu^2)}{2(x-x_\nu)n^4(n-1)^4P_{n-1}^5(x_\nu)}\right|$$
$$\cdot O\left(\frac{1}{n^2\sin^{\frac{1}{2}} t\sin^{\frac{3}{2}} t_\nu} + \frac{\sin^{\frac{1}{2}} t_\nu}{n^2|x-x_\nu|\sin^{\frac{1}{2}} t} + \frac{\sin^{\frac{1}{2}} t}{n^2|x-x_\nu|\sin^{\frac{1}{2}} t_\nu}\right)$$

$$= O\left(\frac{\sin^4 t_\nu}{n^6\sin^2\frac{t-t_\nu}{2}}\right).$$

This completes the proof of the case when $|t-t_\nu| > \frac{c}{n}$.

We obtain the second estimate in the lemma using the estimates

$$|\pi_n'(x)\pi_k(x) - \pi_n(x)\pi_k'(x)| = O\left(n^{\frac{3}{2}}k^{\frac{1}{2}}\right),$$

and

(3.2.17)

$$\sin t = O(\sin t_\nu), \qquad v = 2, \ldots, n-1.$$

Thus, using the above and (3.1.5) and (3.1.9), we have when $|t - t_\nu| \le \frac{c}{n}$

$$\left| \frac{\pi_n^2(x)(1 - x_\nu^2)}{n^4(n-1)^4 P_{n-1}^5(x_\nu)} \right| = O\left(\frac{\sin t \sin^{\frac{9}{2}} t_\nu}{n^{\frac{9}{2}}} \right)$$

and

$$\left| \sum_{k=2}^{n-1} \frac{(2k-1)P_{k-1}'(x_\nu)}{k(k-1)\lambda_k} \Big(\pi_n'(x)\pi_k(x) - \pi_n(x)\pi_k'(x) \Big) \right| = O\left(\frac{n^{\frac{1}{2}}}{\sin^{\frac{3}{2}} t_\nu} \right).$$

Hence the proof of the lemma is complete. □

<u>Lemma 3.2.2</u> We have

$$\sum_{\nu=2}^{n-1} \frac{|E_\nu(x)|}{(1 - x_\nu^2)^2} = O(n^{-4}).$$

<u>Proof</u>

By using Lemma 3.2.1 and (3.1.11)

$$\sum_{\nu=2}^{n-1} \frac{|E_\nu(x)|}{(1 - x_\nu^2)^2} = O\left(n^{-6} \sum_{\nu=2}^{n-1} \frac{1}{\sin t_\nu \sin \frac{|t - t_\nu|}{2}} + n^{-6} \sum_{\nu=2}^{n-1} \frac{1}{\sin^2 \frac{t - t_\nu}{2}} \right) = O(n^{-4}). \square$$

The two sums above can be found in the paper of Szabados and Varma [43], page 744.

3.3 Estimate of the Fundamental Polynomials of the Third Kind

Lemma 3.3.1 We have

(3.3.1)

$$|D_1(x)| = O(n^{-6}), \qquad |D_n(x)| = O(n^{-6}),$$

(3.3.2)

$$\sum_{\nu=2}^{n-1} \frac{|D_\nu(x)|}{1-x_\nu^2} = O\left(\frac{\log n}{n^3}\right),$$

(3.3.3)

$$\sum_{\nu=2}^{n-1} \frac{|D_\nu(x)|}{(1-x_\nu^2)^2} = O(n^{-2}).$$

Proof

We have on using (2.1.1), (2.1.8), (2.1.11), (3.1.4) and (3.1.5)

$$\left|-\frac{\pi_n^3(x)\ell_1(x)}{6n^3(n-1)^3}\right| = O\left(\frac{(1+x)P_{n-1}'(x)\pi_n^3(x)}{6n^4(n-1)^4}\right) = O(n^{-6})$$

and using (3.1.3) and Lemma 3.2.2 we get

$$\left|-4\sum_{k=2}^{n-1} \frac{P_{n-1}^4(x_k)}{(x_k-1)} E_k(x)\right| = O\left(\sum_{k=2}^{n-1} P_{n-1}^4(x_k)\frac{E_k(x)}{1-x_k^2}\right)$$

$$= O\left(n^{-2}\sum_{k=2}^{n-1}\frac{|E_k(x)|}{(1-x_k^2)^2}\right) = O(n^{-6}),$$

so that $|D_1(x)| = O(n^{-6})$. Observing that $D_n(x) = -D_1(-x)$ gives (3.3.1).

We note next that for $|t-t_\nu| \le \frac{c}{n}$, on using (3.1.5), (3.1.9), (3.2.17) and that $|\ell_\nu(x)| \le 1$, we have

(3.3.4)

$$\left|\frac{\pi_n^3(x)\ell_\nu(x)}{6(1-x_\nu^2)n^3(n-1)^3P_{n-1}^3(x_\nu)}\right| = O\left(\frac{(1-x^2)^{\frac{3}{4}}n^{\frac{3}{2}}(1-x_\nu^2)^{\frac{3}{4}}n^{\frac{3}{2}}}{6(1-x_\nu^2)n^3(n-1)^3}\right) = O(n^{-3})$$

and further using (3.1.7)–(3.1.8) we obtain

(3.3.5)

$$\left|\frac{\pi_n^3(x)\ell_\nu(x)}{6(1-x_\nu^2)^2n^3(n-1)^3P_{n-1}^3(x_\nu)}\right| = O\left(\frac{(1-x^2)^{\frac{3}{4}}n^{\frac{3}{2}}(1-x_\nu^2)^{\frac{3}{4}}n^{\frac{3}{2}}}{6(1-x_\nu^2)^2n^3(n-1)^3}\right)$$

$$= O\left(\frac{1}{6(1-x_\nu^2)^{\frac{1}{2}}(n-1)^3}\right) = O(n^{-2}).$$

Now, applying (2.1.8), (2.1.3), (3.1.5), (3.1.9) and (3.1.11) we have

(3.3.6)

$$\left|\sum_{|t-t_\nu|>\frac{c}{n}} -\frac{\pi_n^3(x)\ell_\nu(x)}{6(1-x_\nu^2)n^3(n-1)^3P_{n-1}^3(x_\nu)}\right|$$

$$= \left|\sum_{|t-t_\nu|>\frac{c}{n}} \frac{\pi_n^4(x)}{6(1-x_\nu^2)n^4(n-1)^4P_{n-1}^4(x_\nu)(x-x_\nu)}\right|$$

$$= O\left(\sum_{|t-t_\nu|>\frac{c}{n}} \frac{\sin^2 t}{(n-1)^4\sin\frac{|t-t_\nu|}{2}\sin\frac{t+t_\nu}{2}}\right)$$

$$= O\left(n^{-4}\sum_{|t-t_\nu|>\frac{c}{n}} \frac{\sin t}{\sin\frac{|t-t_\nu|}{2}}\right) = O\left(\frac{\log n}{n^3}\right).$$

Next, using (2.1.8), (2.1.9), (2.1.3) and then (3.1.3), (3.1.9), (3.1.11) and Lemma 3.2.2 we obtain

(3.3.7)

$$\sum_{\nu=2}^{n-1}\left|-\frac{4}{P_{n-1}^3(x_\nu)(1-x_\nu^2)}\sum_{k=2}^{n-1}P_{n-1}^3(x_k)\ell_\nu'(x_k)E_k(x)\right|$$

$$= 4\sum_{\nu=2}^{n-1}\left|\sum_{|t_k-t_\nu|>\frac{c}{n}} \frac{P_{n-1}^4(x_k)}{(1-x_\nu^2)(x_k-x_\nu)P_{n-1}^4(x_\nu)}E_k(x)\right|$$

$$= O\left(4\sum_{k=2}^{n-1}\left(\frac{|E_k(x)|}{(1-x_k^2)^2}\sum_{|t_k-t_\nu|>\frac{c}{n}}\frac{(1-x_k^2)^2P_{n-1}^4(x_k)}{(1-x_\nu^2)(x_k-x_\nu)P_{n-1}^4(x_\nu)}\right)\right)$$

$$= O\left(\sum_{k=2}^{n-1}\left(\frac{|E_k(x)|}{(1-x_k^2)^2}\sum_{|t_k-t_\nu|>\frac{c}{n}}\frac{\sin t_k}{\sin\frac{|t_k-t_\nu|}{2}}\right)\right)$$

$$= O\left(n\log n\sum_{k=2}^{n-1}\frac{|E_k(x)|}{(1-x_k^2)^2}\right) = O\left(\frac{\log n}{n^3}\right).$$

We note that when $\nu = k$, the double sum in (3.3.7) is 0 as $\ell_\nu'(x_\nu) = 0$, $\nu = 2, \ldots, n-1$. Observing (3.3.4), (3.3.6) and (3.3.7), we see that (3.3.2) holds. To show (3.3.3), we observe (3.3.5), and argue in a manner similar to (3.3.6) and (3.3.7) to obtain

$$\sum_{\nu=2}^{n-1}\left|-\frac{\pi_n^3(x)\ell_\nu(x)}{6(1-x_\nu^2)^2n^3(n-1)^3P_{n-1}^3(x_\nu)}\right| = O\left(n^{-4}\sum_{\nu=2}^{n-1}\frac{1}{\sin t_\nu\sin\frac{|t-t_\nu|}{2}}\right) = O\left(n^{-2}\right)$$

and

$$\sum_{\nu=2}^{n-1}\left|-\frac{4}{(1-x_\nu^2)^2P_{n-1}^3(x_\nu)}\sum_{k=2}^{n-1}P_{n-1}^3(x_k)\ell_\nu'(x_k)E_k(x)\right|$$

$$= O\left(\sum_{k=2}^{n-1}\left(\frac{|E_k(x)|}{(1-x_k^2)^2}\sum_{|t_k-t_\nu|>\frac{c}{n}}\frac{1}{\sin t_\nu\sin\frac{|t_k-t_\nu|}{2}}\right)\right) = O(n^{-2}).$$

This completes the proof of the lemma. □

3.4 Estimate of the Fundamental Polynomials $C_1(x)$ and $C_n(x)$

Lemma 3.4.1 We have

$$|C_1(x)| = O(n^{-4}), \qquad |C_n(x)| = O(n^{-4}).$$

Proof

Using (2.1.3), (2.1.8), (2.1.11) and (3.1.5) we obtain

$$\left|\frac{(1-x)\pi_n^2(x)\ell_1^2(x)}{4n(n-1)}\right| = \left|\frac{(1-x)\pi_n^4(x)}{4n(n-1)(x-1)^2\pi_n'(1)^2}\right|$$

$$= O\left(\frac{(1-x)(1-x^2)n^2}{4n^3(n-1)^3(1-x)^2}\right) = O\left(n^{-4}\right).$$

Using (2.1.8), (2.1.11), (3.1.5), Markov's and Bernstein's inequalities

$$\left|\frac{\pi_n^2(x)\ell_1^2(x)}{2n^2(n-1)^2}\right| = \left|\frac{\pi_n^4(x)}{2n^2(n-1)^2(x-1)^2\pi_n'(1)^2}\right|$$

$$= O\left(\frac{(1-x^2)^2P_{n-1}'(x)P_{n-1}'(x)\pi_n^2(x)}{2n^4(n-1)^4(x-1)^2}\right)$$

$$= O\left(\frac{(1+x)^2n^2\frac{n}{(1-x^2)^{\frac{1}{2}}}(1-x^2)^{\frac{1}{2}}n}{2n^4(n-1)^4}\right) = O(n^{-4}),$$

and finally, using (2.1.9), (2.1.3), (3.1.3), $|P_{n-1}(x)| \leq 1$ and Lemma 3.2.2

$$\left|-12\sum_{k=2}^{n-1}P_{n-1}^2(x_k)\ell_1'(x_k)^2\Big(1-\frac{n(n-1)}{2}(x_k-1)\Big)E_k(x)\right|$$

$$= O\left(\sum_{k=2}^{n-1}P_{n-1}^2(x_k)\ell_1'(x_k)^2E_k(x)\right) + O\left(n(n-1)\sum_{k=2}^{n-1}(x_k-1)P_{n-1}^2(x_k)\ell_1'(x_k)^2E_k(x)\right)$$

$$= O\left(\sum_{k=2}^{n-1}\frac{P_{n-1}^4(x_k)}{(x_k-1)^2}E_k(x)\right) + O\left(n(n-1)\sum_{k=2}^{n-1}\frac{P_{n-1}^4(x_k)}{x_k-1}E_k(x)\right)$$

$$= O\left(\sum_{k=2}^{n-1} \frac{|E_k(x)|}{(1-x_k^2)^2} + n(n-1)\sum_{k=2}^{n-1} \frac{1}{n^2(1-x_k^2)}\frac{1}{1-x_k}E_k(x)\right)$$

$$= O\left(\sum_{k=2}^{n-1} \frac{E_k(x)}{(1-x_k^2)^2}\right) = O(n^{-4}).$$

Noting that $C_n(x) = C_1(-x)$ completes the proof. □

3.5 Estimate of the Fundamental Polynomials of the Second Kind

Lemma 3.5.1 We have

(3.5.1)

$$|B_1(x)| = O(n^{-2}), \qquad |B_n(x)| = O(n^{-2}),$$

(3.5.2)

$$\sum_{\nu=2}^{n-1} \frac{|B_\nu(x)|}{(1-x_\nu^2)^{\frac{1}{2}}} = O\left(\frac{\log n}{n}\right).$$

Proof

We show first $|B_1(x)| = O(n^{-2})$. Applying (2.1.8), (2.1.11), (2.1.1), Bernstein's inequality and $\ell_1^2(x) \le 1$ we obtain

$$\left|-\frac{\pi_n(x)\ell_1^3(x)}{n(n-1)}\right| = \left|\frac{\pi_n^2(x)\ell_1^2(x)}{(x-1)n^2(n-1)^2}\right| = O\left(\frac{(1-x^2)P'_{n-1}(x)\pi_n(x)\ell_1^2(x)}{(x-1)n^2(n-1)^2}\right)$$

$$= O\left(\frac{(1+x)\ell_1^2(x)}{n^2(n-1)^2}(1-x^2)P'_{n-1}(x)^2\right) = O(n^{-2}).$$

Next, applying Markov's inequality (to $\ell_1'(x_k)$), (2.1.8), (2.1.3), (3.1.3), $|P_{n-1}(x)| \le 1$ and Lemma 3.2.2 yield

$$\left|\frac{24}{n(n-1)}\sum_{k=2}^{n-1} \pi_n'(x_k)\ell_1'(x_k)^3 E_k(x)\right| = O\left(\sum_{k=2}^{n-1} \pi_n'(x_k)\ell_1'(x_k)^2 E_k(x)\right)$$

$$= O\left(n^2\sum_{k=2}^{n-1}\frac{P_{n-1}^3(x_k)}{(1-x_k)^2}E_k(x)\right) = O\left(n^2\sum_{k=2}^{n-1}\frac{|E_k(x)|}{(1-x_k^2)^2}\right) = O(n^{-2}).$$

Observing $B_n(x) = -B_1(-x)$, we deduce that (3.5.1) holds.

Using (2.1.1), (3.1.9), (3.1.4), (3.2.17) and $|\ell_\nu^3(x)| \le 1$, we note that for $|t-t_\nu| \le \frac{c}{n}$

$$\left|-\frac{\pi_n(x)\ell_\nu^3(x)}{(1-x_\nu^2)^{\frac{1}{2}}n(n-1)P_{n-1}(x_\nu)}\right| = O\left(\frac{(1-x^2)P'_{n-1}(x)}{(1-x_\nu^2)^{\frac{1}{4}}n^{\frac{1}{2}}(n-1)}\right)$$

$$= O\left(\frac{(1-x^2)^{\frac{3}{4}}P'_{n-1}(x)}{n^{\frac{1}{2}}(n-1)}\right) = O(n^{-1}).$$

Observing (2.3.11), (3.3.2) and (3.1.7)–(3.1.8) we have

$$\sum_{\nu=2}^{n-1}\frac{4n(n-1)}{1-x_\nu^2}|D_\nu(x)| = O\left(n^2\sum_{\nu=2}^{n-1}\frac{|D_\nu(x)|}{1-x_\nu^2}\right) = O\left(\frac{\log n}{n}\right),$$

and

$$\left|\sum_{\nu=2}^{n-1}\frac{13n(n-1)x_\nu}{(1-x_\nu^2)^{\frac{1}{2}}(1-x_\nu^2)^2}E_\nu(x)\right| = O\left(n^2\sum_{\nu=2}^{n-1}\frac{|E_\nu(x)|}{(1-x_\nu^2)^{\frac{5}{2}}}\right)$$

$$= O\left(n^3\sum_{\nu=2}^{n-1}\frac{|E_\nu(x)|}{(1-x_\nu^2)^2}\right) = O(n^{-1}).$$

Now, applying (2.1.3), (2.1.8), (3.1.5), (3.1.9), (3.1.11) and $\ell_\nu^2(x) \le 1$ we obtain

$$\left|\sum_{|t-t_\nu|>\frac{c}{n}} -\frac{\pi_n(x)\ell_\nu^3(x)}{(1-x_\nu^2)^{\frac{1}{2}}n(n-1)P_{n-1}(x_\nu)}\right|$$

$$= O\left(\sum_{|t-t_\nu|>\frac{c}{n}}\frac{\pi_n^2(x)\ell_\nu^2(x)}{\sin t_\nu n^2(n-1)^2P_{n-1}^2(x_\nu)|x-x_\nu|}\right)$$

$$= O\left(\sum_{|t-t_\nu|>\frac{c}{n}} \frac{\sin t}{(n-1)^2|x-x_\nu|}\right) = O\left(n^{-2}\sum_{|t-t_\nu|>\frac{c}{n}} \frac{1}{\sin\frac{|t-t_\nu|}{2}}\right) = O\left(\frac{\log n}{n}\right),$$

and further using (2.1.9) and (3.1.3) yield

$$\sum_{\nu=2}^{n-1}\left|\frac{24}{(1-x_\nu^2)^{\frac{1}{2}}}\sum_{k=2}^{n-1}\frac{\pi_n'(x_k)\ell_\nu'(x_k)^3}{\pi_n'(x_\nu)}E_k(x)\right|$$

$$= O\left(\sum_{k=2}^{n-1}\left|\frac{E_k(x)}{(1-x_k^2)^2}\sum_{|t_k-t_\nu|>\frac{c}{n}}\frac{P_{n-1}^4(x_k)(1-x_k^2)^2}{(1-x_\nu^2)^{\frac{1}{2}}|x_k-x_\nu|^3P_{n-1}^4(x_\nu)}\right|\right)$$

$$= O\left(\sum_{k=2}^{n-1}\left(\frac{|E_k(x)|}{(1-x_k^2)^2}\sum_{|t_k-t_\nu|>\frac{c}{n}}\frac{1}{\sin^3\frac{|t_k-t_\nu|}{2}}\right)\right)$$

$$= O\left(n^3\sum_{k=2}^{n-1}\frac{|E_k(x)|}{(1-x_k^2)^2}\right) = O(n^{-1}).$$

This completes the proof of the lemma. □

3.6 Estimate of the Fundamental Polynomials of the First Kind

Lemma 3.6.1 We have

$$\sum_{\nu=1}^{n}|A_\nu(x)| = O(1).$$

Proof

We show first that $|A_1(x)| = O(1)$. It suffices to note that on using Bernstein's inequality, $|\ell_1(x)| \leq 1$ and Lemma 3.2.2

$$\left|\sum_{k=2}^{n-1}\ell_1'(x_k)^4E_k(x)\right| = O\left(n^4\sum_{k=2}^{n-1}\frac{|E_k(x)|}{(1-x_k^2)^2}\right) = O(1),$$

and observing $A_n(x) = A_1(-x)$, we deduce $|A_n(x)| = O(1)$. Now, as $\ell_\nu^4(x) \le \ell_\nu^2(x)$, we have

$$\sum_{\nu=2}^{n-1} \ell_\nu^4(x) \le \sum_{\nu=2}^{n-1} \ell_\nu^2(x) \le 1,$$

and observing (3.3.2), (3.1.7)-(3.1.8) and Lemma 3.2.2 we have

$$\left|\sum_{\nu=2}^{n-1} \frac{4n(n-1)x_\nu}{1-x_\nu^2} D_\nu(x)\right| = O\left(n^2 \sum_{\nu=2}^{n-1} \frac{|D_\nu(x)|}{1-x_\nu^2}\right) = O\left(\frac{\log n}{n}\right),$$

and

$$\sum_{\nu=2}^{n-1} \left|\frac{24n(n-1)}{5(1-x_\nu^2)^2}\left(n^2 - n + 3 - \frac{4}{1-x_\nu^2}\right) E_\nu(x)\right| = O\left(n^4 \sum_{\nu=2}^{n-1} \frac{|E_\nu(x)|}{(1-x_\nu^2)^2} + n^2 \sum_{\nu=2}^{n-1} \frac{|E_\nu(x)|}{(1-x_\nu^2)^3}\right)$$

$$= O\left(n^4 \sum_{\nu=2}^{n-1} \frac{|E_\nu(x)|}{(1-x_\nu^2)^2}\right) = O(1).$$

It remains only to estimate

$$\left|\sum_{\nu=2}^{n-1} \Big(-24 \sum_{\nu=2}^{n-1} \ell_\nu'(x_k)^4 E_k(x)\Big)\right|,$$

and applying (2.1.9), (2.1.3), (3.1.3), (3.1.9) and (3.1.11) yields

$$O\left(\sum_{k=2}^{n-1} \Big(\frac{|E_k(x)|}{(1-x_k^2)^2} \sum_{|t_k - t_\nu| > \frac{c}{n}} \frac{(1-x_k^2)^2 P_{n-1}^4(x_k)}{(x_k - x_\nu)^4 P_{n-1}^4(x_\nu)}\Big)\right)$$

$$= O\left(\sum_{k=2}^{n-1} \Big(\frac{|E_k(x)|}{(1-x_k^2)^2} \sum_{|t_k - t_\nu| > \frac{c}{n}} \frac{(1-x_k^2)(1-x_\nu^2)}{|x_k - x_\nu|^4}\Big)\right)$$

$$= O\left(\sum_{k=2}^{n-1}\left(\frac{|E_k(x)|}{(1-x_k^2)^2}\sum_{|t_k-t_\nu|>\frac{c}{n}}\frac{\sin^2 t_k\sin^2 t_\nu}{\sin^4\frac{t_k-t_\nu}{2}\sin^4\frac{t_k+t_\nu}{2}}\right)\right)$$

$$= O\left(\sum_{k=2}^{n-1}\left(\frac{|E_k(x)|}{(1-x_k^2)^2}\sum_{|t_k-t_\nu|>\frac{c}{n}}\frac{1}{\sin^4\frac{t_k-t_\nu}{2}}\right)\right)$$

$$= O\left(n^4\sum_{k=2}^{n-1}\frac{|E_k(x)|}{(1-x_k^2)^2}\right) = O(1).$$

This completes the proof of the lemma. □

3.7 Proof of the Convergence Theorem

Denote by $[x]$ the largest integer less than or equal to x. This expression is known as the greatest integer in x. Let $f \in C[-1,1]$, $m = \left[\frac{n}{\log n}\right]$, and consider polynomials $p_m(x)$ of degree at most m such that

(3.7.1)

$$\|f^{(j)}(x) - p_m^{(j)}(x)\| = O(m^j)w_4(f, \frac{1}{m})$$

and

(3.7.2)

$$\|p_m^{(j)}(x)\| = O(m^j)w_j(f, \frac{1}{m}), \qquad 1 \le j \le 4$$

The above polynomials exist by a paper of H. Gonska [16], page 165.

Since 'modified' $(0,1,3,4)$ interpolation is uniquely determined, we have

$$p_m(x) - R_n(p_m, x) = \sum_{\nu=1}^{n} p_m'(x_\nu)B_\nu(x) + p_m''(1)C_1(x) + p_m''(-1)C_n(x) + \sum_{\nu=1}^{n} p_m'''(x_\nu)D_\nu(x)$$

$$+\sum_{\nu=2}^{n-1} p_m^{(4)}(x_\nu)E_\nu(x).$$

Thus, using (3.7.2) and Lemmas 3.5.1, 3.4.1, 3.3.1 and 3.2.2 we have

$$|p_m(x)-R_n(p_m,x)| = \sum_{\nu=1}^{n} p'_m(x_\nu)B_\nu(x)+p''_m(1)C_1(x)+p''_m(-1)C_n(x)+\sum_{\nu=1}^{n} p'''_m(x_\nu)D_\nu(x)$$

$$+\sum_{\nu=2}^{n-1} p^{(4)}_m(x_\nu)E_\nu(x)$$

(3.7.3)

$$= O(m)w_1(f,\frac{1}{m})\sum_{\nu=1}^{n}|B_\nu(x)| + O(m^2)w_2(f,\frac{1}{m})(|C_1(x)|+|C_n(x)|)$$

$$+O(m^3)w_3(f,\frac{1}{m})\sum_{\nu=1}^{n}|D_\nu(x)| + O(m^4)w_4(f,\frac{1}{m})\sum_{\nu=2}^{n-1}|E_\nu(x)|.$$

Now, from (3.7.1), (3.7.3) and Lemma 3.6.1 we get

$$\begin{aligned}
\|f-R_n(f)\| &\le \|f-p_m\| + \|p_m - R_n(p_m)\| + \|R_n(p_m-f)\| \\
&\le \|f-p_m\| + \|p_m - R_n(p_m)\| + \|p_m - f\| \; \|\sum_{\nu=1}^{n}|A_\nu(x)|\,\| \\
&= O\left(w_4\Big(f,\frac{\log n}{n}\Big)\right) + O\left(w_1\Big(f,\frac{\log n}{n}\Big)\right) + O\left(w_4\Big(f,\frac{\log n}{n}\Big)\right) \\
&= O\left(w_1\Big(f,\frac{\log n}{n}\Big)\right).
\end{aligned}$$

This proves the theorem. □

CHAPTER 4
ERDÖS-TYPE INEQUALITIES

4.1 Main Results

Let us denote by S_n the set of all polynomials whose degree is n and whose zeros are all real and lie outside $(-1,1)$, and denote by L_n the set of all polynomials of the form

(4.1.1)
$$P_n(x) = \sum_{k=0}^{n} a_k q_{nk}(x), \quad a_k \geq 0 \quad (k = 0, 1, \ldots, n)$$

where $q_{nk}(x) = (1+x)^{n-k}(1-x)^k$.

Here we present two theorems concerned with finding a uniform upper bound for the expression

$$\frac{\int_{-1}^{1} \omega(x)(P_n'(x))^p dx}{\int_{-1}^{1} \omega(x)(P_n(x))^p dx}$$

where $\omega(x) = (1-x^2)^\alpha, \alpha > -1$, when $p = 2$, and $\omega(x) = (1-x^2)^3$, when $p = 4$, and where the polynomials P_n are restricted to the Lorentz class L_n of polynomials.

It is known [22] that if $P_n \in S_n$, then $P_n \in L_n$ or $-P_n \in L_n$. Thus, Theorem 4.1 is an extension of the classical theorem of P. Erdös [10] for P_n in S_n, as well as the results of P. Erdös and A.K. Varma [11] and the Theorem 3.4 in Milovanović and Petković [28], in the L^2 norm with the ultraspherical weight $\omega(x) = (1-x^2)^\alpha$, $\alpha > -1$. In Theorem 4.2 we present the first sharp extension of the inequality of Erdös in a weighted L^4 norm. Note that Theorems 4.1 and 4.2 provide the polynomials which attain the given upper bounds.

We shall later see that for each n there exists a unique positive solution to the equation

$$2\alpha^4+(8n-5)\alpha^3+(12n^2-17n+4)\alpha^2+(8n^3-20n^2+11n-1)\alpha-2n(2n^2-5n+4)=0.$$

Denote this solution by α_n.

Theorem 4.1 Let $P_n \in L_n$, $n \geq 2$ and $\alpha > -1$ real. Then we have for $\alpha \geq \alpha_n$

$$\int_{-1}^{1}(1-x^2)^{\alpha}(P_n'(x))^2dx \leq \frac{n^2(2n+2\alpha+1)(n+\alpha)}{2(2n+\alpha)(2n+\alpha-1)}\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx$$

with equality for $P_n(x)=c(1\pm x)^n$. For $-1<\alpha\leq\alpha_n$ the inequality becomes

$$\int_{-1}^{1}(1-x^2)^{\alpha}(P_n'(x))^2dx$$
$$\leq \frac{(2n+2\alpha+1)(n+\alpha)[\alpha(\alpha-1)n^2+2(n-1)(n-(\alpha-1)(2\alpha-1))]}{2(\alpha+1)(\alpha+2)(2n+\alpha-2)(2n+\alpha-3)}$$
$$\times\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx$$

with equality for $P_n(x)=c(1\pm x)^{n-1}(1\mp x)$.

Theorem 4.2 Let $P_n \in L_n$. Then we have

$$\int_{-1}^{1}(1-x^2)^3(P_n'(x))^4dx \leq \frac{n^3(4n+7)(4n+6)(4n+5)(4n+4)}{64(4n+3)(4n+2)(4n+1)}\int_{-1}^{1}(1-x^2)^3(P_n(x))^4dx$$

with equality if and only if $P_n(x)=c(1\pm x)^n$.

4.2 Some Lemmas

Lemma 4.2.1 Let $P_n \in L_n$ and $\alpha > 0$ real. Then we have

(4.2.1)

$$\int_{-1}^{1}(1-x^2)^{\alpha-1}(P_n(x))^2dx \leq \frac{(2n+2\alpha+1)(n+\alpha)}{2\alpha(2n+\alpha)}\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx$$

with equality if and only if $P_n(x) = c(1 \pm x)^n$. In the case $P_n \in L_n$ and $P_n(\pm 1) = 0$, the inequality becomes

$$(4.2.2)\ \int_{-1}^{1}(1-x^2)^{\alpha-1}(P_n(x))^2dx \le \frac{(2n+2\alpha+1)(n+\alpha)}{2(\alpha+2)(2n+\alpha-2)}\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx$$

with equality if and only if $P_n(x) = c(1 \pm x)^{n-1}(1 \mp x)$.

Proof

We write $Q_{2n}(x) = (P_n(x))^2$. Then $Q_{2n} \in L_{2n}$ and we have from (4.1.1)

$$Q_{2n}(x) = \sum_{k=0}^{2n} a_k(1+x)^{2n-k}(1-x)^k.$$

Thus,

(4.2.3)

$$\frac{\int_{-1}^{1}(1-x^2)^{\alpha-1}(P_n(x))^2dx}{\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx} = \frac{\sum_{k=0}^{2n} a_k \int_{-1}^{1}(1+x)^{2n+\alpha-1-k}(1-x)^{k+\alpha-1}dx}{\sum_{k=0}^{2n} a_k \int_{-1}^{1}(1+x)^{2n+\alpha-k}(1-x)^{k+\alpha}dx},$$

and we use the known formula

(4.2.4)

$$\int_{-1}^{1}(1+x)^p(1-x)^q dx = \frac{\Gamma(p+1)\Gamma(q+1)2^{p+q+1}}{\Gamma(p+q+2)}$$

to obtain

$$\frac{\int_{-1}^{1}(1+x)^{2n+\alpha-1-k}(1-x)^{k+\alpha-1}dx}{\int_{-1}^{1}(1+x)^{2n+\alpha-k}(1-x)^{k+\alpha}dx} = \frac{\frac{\Gamma(2n+\alpha-k)\Gamma(k+\alpha)2^{2n+2\alpha-1}}{\Gamma(2n+2\alpha)}}{\frac{\Gamma(2n+\alpha+1-k)\Gamma(k+\alpha+1)2^{2n+2\alpha+1}}{\Gamma(2n+2\alpha+2)}}$$

$$= \frac{(2n+2\alpha+1)(2n+2\alpha)}{4(k+\alpha)(2n+\alpha-k)} \le \frac{(2n+2\alpha+1)(n+\alpha)}{2\alpha(2n+\alpha)}$$

for $k = 0, 1, 2, \ldots, 2n$, where equality holds iff $k = 0$ or $k = 2n$. Applying the above to (4.2.3) yields (4.2.1). If $P_n(\pm 1) = 0$, then we note that k runs only from 2 to $2n-2$ above. Employing this observation in (4.2.3) proves (4.2.2). This completes the proof of the lemma. $\square$

Lemma 4.2.2 Let $P_n \in L_n$ and $\alpha \geq 1$ real. Then we have

(4.2.5)

$$\int_{-1}^{1} (1-x^2)^{\alpha}(P_n'(x))^2 dx \leq \frac{\alpha n^2}{2n+\alpha-1} \int_{-1}^{1} (1-x^2)^{\alpha-1}(P_n(x))^2 dx$$

with equality if and only if $P_n(x) = c(1 \pm x)^n$. In the case that $P_n \in L_n$ and $P_n(\pm 1) = 0$, we have for $\alpha > -1$ real the inequality

(4.2.6)

$$\int_{-1}^{1} (1-x^2)^{\alpha}(P_n'(x))^2 dx \leq \frac{\alpha(\alpha-1)n^2 + 2(n-1)(n-(\alpha-1)(2\alpha-1))}{(\alpha+1)(2n+\alpha-3)}$$

$$\times \int_{-1}^{1} (1-x^2)^{\alpha-1}(P_n(x))^2 dx$$

with equality if and only if $P_n(x) = c(1 \pm x)^{n-1}(1 \mp x)$.

Proof

From (4.1.1) we have

$$(P_n(x))^2 = \sum_{j=0}^{n}\sum_{k=0}^{n} a_k a_j q_{nk}(x) q_{nj}(x)$$

so that we may write

$$\int_{-1}^{1} (1-x^2)^{\alpha-1}(P_n(x))^2 dx = \sum_{j=0}^{n}\sum_{k=0}^{n} a_k a_j \int_{-1}^{1} (1-x^2)^{\alpha-1} q_{nk}(x) q_{nj}(x) dx$$

$$= \sum_{j=0}^{n}\sum_{k=0}^{n} a_k a_j \int_{-1}^{1} (1+x)^{2n+\alpha-1-k-j}(1-x)^{k+j+\alpha-1}\, dx.$$

On using (4.2.4) and writing $\ell = k+j$ we obtain

(4.2.7)

$$\int_{-1}^{1} (1-x^2)^{\alpha-1}(P_n(x))^2 dx = \sum_{j=0}^{n}\sum_{k=0}^{n} \frac{a_k a_j \Gamma(2n+\alpha-\ell)\Gamma(\ell+\alpha) 2^{2n+2\alpha-1}}{\Gamma(2n+2\alpha)}.$$

Next we show that for $\alpha \geq 1$

$$\int_{-1}^{1} (1-x^2)^\alpha (P_n'(x))^2 dx = \sum_{j=0}^{n} \sum_{k=0}^{n} a_k a_j \int_{-1}^{1} (1-x^2)^\alpha q_{nk}'(x) q_{nj}'(x) dx$$

(4.2.8)

$$\leq \frac{\alpha n^2}{2n+\alpha-1} \sum_{j=0}^{n} \sum_{k=0}^{n} \frac{a_k a_j \Gamma(2n+\alpha-\ell)\Gamma(\ell+\alpha) 2^{2n+2\alpha-1}}{\Gamma(2n+2\alpha)}.$$

As $q_{nk}(x) = (1+x)^{n-k}(1-x)^k$, we have

$$q_{nk}'(x) = (n-k)(1+x)^{n-k-1}(1-x)^k - k(1-x)^{n-k}(1-x)^{k-1},$$

and using (4.2.4) yields

(4.2.9)

$$I_{k,j} = \int_{-1}^{1} (1-x^2)^\alpha q_{nk}'(x) q_{nj}'(x) dx = \frac{2^{2n+2\alpha-1}}{\Gamma(2n+2\alpha)} \Big[(n-k)(n-j)\Gamma(2n+\alpha-1-\ell)\Gamma(\ell+\alpha+1)$$

$$+kj\Gamma(2n+\alpha+1-\ell)\Gamma(\ell+\alpha-1) - (n\ell-2kj)\Gamma(2n+\alpha-\ell)\Gamma(\ell+\alpha) \Big]$$

$$= \frac{\Gamma(2n+\alpha-\ell)\Gamma(\ell+\alpha)2^{2n+2\alpha-1}}{\Gamma(2n+2\alpha)}$$

$$\times \left\{ \frac{(n-k)(n-j)(\ell+\alpha)}{2n+\alpha-1-\ell} + \frac{kj(2n+\alpha-\ell)}{\ell+\alpha-1} - n\ell + 2kj \right\}.$$

We denote the portion in brackets by μ_{kj} and simplify the expression as follows, denoting $\ell = k+j$ and later using that $4kj = \ell^2 - (k-j)^2$. We have

$$\mu_{kj} = \frac{(n^2 - n\ell + kj)(\ell+\alpha)}{2n+\alpha-1-\ell} + \frac{kj(2n+\alpha-\ell)}{\ell+\alpha-1} - n\ell + 2kj$$

$$= \frac{\alpha n^2 + n^2\ell}{2n+\alpha-1-\ell} - \frac{n\ell(2n+2\alpha-1)}{2n+\alpha-1-\ell} + \frac{kj(2n+2\alpha-1)(2n+2\alpha-2)}{(2n+\alpha-1-\ell)(\ell+\alpha-1)}$$

$$= \frac{\alpha n^2}{2n+\alpha-1-\ell} - \frac{\ell[2n^2+(2n-\ell)(2\alpha^2-3\alpha+1-n)]}{2(2n+\alpha-1-\ell)(\ell+\alpha-1)}$$

$$-\frac{(k-j)^2(2n+2\alpha-1)(n+\alpha-1)}{2(2n+\alpha-1-\ell)(\ell+\alpha-1)}$$

$$= \frac{2\alpha(\alpha-1)n^2+\ell(2n-\ell)(n-(\alpha-1)(2\alpha-1))-(k-j)^2(2n+2\alpha-1)(n+\alpha-1)}{2(2n+\alpha-1-\ell)(\ell+\alpha-1)}.$$

We show that $\frac{\alpha n^2}{2n+\alpha-1} - \mu_{kj} \geq 0$ for $k, j = 0, 1, \ldots, n, (\alpha \geq 1)$ and apply this to (4.2.9) to obtain (4.2.8). We have

$$\frac{\alpha n^2}{2n+\alpha-1} - \mu_{kj} = \frac{\alpha n^2}{2n+\alpha-1}$$

$$-\frac{2\alpha(\alpha-1)n^2+\ell(2n-\ell)(n-(\alpha-1)(2\alpha-1))-(k-j)^2(2n+2\alpha-1)(n+\alpha-1)}{2(2n+\alpha-1-\ell)(\ell+\alpha-1)}$$

$$\geq \frac{\ell(2n-\ell)[2(\alpha-1)n^2+(4\alpha-3)(\alpha-1)n+(\alpha-1)^2(2\alpha-1)]}{2(2n+\alpha-1)(2n+\alpha-1-\ell)(\ell+\alpha-1)} \geq 0,$$

with equality iff $\ell = 0$ or $\ell = 2n$.

Observing (4.2.8) and (4.2.7) we have (4.2.5). Lastly, for the case $P_n(\pm 1) = 0$ $(k, j = 1, 2, \ldots, n-1)$ and $-1 < \alpha < 1$ we have

$$\frac{\alpha(\alpha-1)n^2+2(n-1)(n-(\alpha-1)(2\alpha-1))}{(\alpha+1)(2n+\alpha-3)} - \mu_{kj}$$

$$= \frac{(2n-2(n-1)\alpha-\alpha^2)[\ell(2n-\ell)-4(n-1)]}{(\alpha+1)(2n+\alpha-3)(\ell+\alpha-1)(2n+\alpha-1-\ell)}$$
$$+\frac{(k-j)^2(2n+2\alpha-1)(n+\alpha-1)}{(\ell+\alpha-1)(2n+\alpha-1-\ell)} \geq 0,$$

as $2n-2(n-1)\alpha-\alpha^2>1$ for $-1<\alpha<1$, and $\ell(2n-\ell)-4(n-1)\geq 0$, with equality iff $k=j=1$ or $k=j=n-1$. This yields

$$\int_{-1}^{1}(1-x^2)^{\alpha}(P_n'(x))^2dx$$

$$\leq \frac{\alpha(\alpha-1)n^2+2(n-1)(n-(\alpha-1)(2\alpha-1))}{(\alpha+1)(2n+\alpha-3)} \times \sum_{j=0}^{n}\sum_{k=0}^{n}\frac{a_k a_j\Gamma(2n+\alpha-\ell)\Gamma(\ell+\alpha)2^{2n+2\alpha-1}}{\Gamma(2n+2\alpha)}.$$

Combining the above with (4.2.7) yields (4.2.6), completing the proof of Lemma 4.2.2.

□

4.3 Proofs of Theorems

Proof of Theorem 4.1

Let $P_n\in L_n$. The case $\alpha\geq 1$ follows from (4.2.5) and (4.2.1). Now, let $n\geq 2$ and $-1<\alpha<1$. We write

$$P_n(x)=a_0(1+x)^n+Q_n(x)+a_n(1-x)^n$$

where $Q_n\in L_n$ and $Q_n(\pm 1)=0$. Then

(4.3.1)

$$\begin{aligned}\int_{-1}^{1}(1-x^2)^{\alpha}(P_n'(x))^2dx &= n^2a_0^2\int_{-1}^{1}(1-x^2)^{\alpha}(1+x)^{2n-2}dx\\ &+n^2a_n^2\int_{-1}^{1}(1-x^2)^{\alpha}(1-x)^{2n-2}dx+\int_{-1}^{1}(1-x^2)^{\alpha}(Q_n'(x))^2dx\\ &-2n^2a_0a_n\int_{-1}^{1}(1-x^2)^{n+\alpha-1}dx+2na_0\int_{-1}^{1}(1-x^2)^{\alpha}(1+x)^{n-1}Q_n'(x)dx\\ &-2na_n\int(1-x^2)^{\alpha}(1-x)^{n-1}Q_n'(x)dx.\end{aligned}$$

We show the last 2 integrals are nonpositive. Integrating by parts we obtain

$$I_1 = 2na_0 \int_{-1}^{1} (1-x^2)^\alpha (1+x)^{n-1} Q_n'(x)dx = 2na_0 \int_{-1}^{1} (1-x)^\alpha (1+x)^{n+\alpha-1} Q_n'(x)dx$$

$$= -2n(n+\alpha-1)a_0 \int_{-1}^{1} (1+x)^{n+\alpha-2}(1-x)^\alpha Q_n(x)dx$$

$$+2n\alpha a_0 \int_{-1}^{1} (1+x)^{n+\alpha-1}(1-x)^{\alpha-1} Q_n(x)dx.$$

Thus, we have $I_1 \le 0$ for $-1 < \alpha \le 0$, and if for $0 < \alpha < 1$ we show

(4.3.2)
$$\frac{\int_{-1}^{1}(1+x)^{n+\alpha-1}(1-x)^{\alpha-1}Q_n(x)dx}{\int_{-1}^{1}(1+x)^{n+\alpha-2}(1-x)^{\alpha}Q_n(x)dx} \le \frac{2n+\alpha-2}{2\alpha},$$

then we will have

$$I_1 \le -2n(n+\alpha-1)a_0 \int_{-1}^{1} (1+x)^{n+\alpha-2}(1-x)^\alpha Q_n(x)dx$$

$$+(2n+\alpha-2)na_0 \int_{-1}^{1} (1+x)^{n+\alpha-2}(1-x)^\alpha Q_n(x)dx$$

$$= -\alpha a_0 n \int_{-1}^{1} (1-x)^\alpha (1+x)^{n+\alpha-2} Q_n(x)dx \le 0,$$

as desired. We show now (4.3.2). It suffices to consider

$$\frac{\int_{-1}^{1}(1+x)^{n+\alpha-1}(1-x)^{\alpha-1}q_{nk}(x)dx}{\int_{-1}^{1}(1+x)^{n+\alpha-2}(1-x)^{\alpha}q_{nk}(x)dx}$$

$$= \frac{\int_{-1}^{1}(1+x)^{2n+\alpha-1-k}(1-x)^{k+\alpha-1}dx}{\int_{-1}^{1}(1+x)^{2n+\alpha-2-k}(1-x)^{k+\alpha}dx} = \frac{2n+\alpha-1-k}{k+\alpha}$$

$$\le \frac{2n+\alpha-2}{\alpha+1} \quad (k = 1, 2, \ldots, n-1)$$

$$\leq \frac{2n+\alpha-2}{2\alpha} \quad (0<\alpha\leq 1)$$

showing (4.3.2). In the same manner we obtain

$$I_2 = -2na_n \int_{-1}^{1} (1-x^2)^\alpha (1-x)^{n-1} Q_n'(x)dx = -2na_n \int_{-1}^{1} (1-x)^{n+\alpha-1}(1+x)^\alpha Q_n'(x)dx$$

$$= -2n(n+\alpha-1)a_n \int_{-1}^{1} (1+x)^\alpha (1-x)^{n+\alpha-2} Q_n(x)dx$$
$$+2n\alpha a_n \int_{-1}^{1} (1+x)^{\alpha-1}(1-x)^{n+\alpha-1} Q_n(x)dx,$$

showing that $I_2 \leq 0$ for $-1<\alpha<1$. So from the above, (4.3.1) and (4.2.4) we obtain

$$\int_{-1}^{1} (1-x^2)^\alpha (P_n'(x))^2 dx \leq n^2(a_0^2+a_n^2)\frac{\Gamma(2n+\alpha-1)\Gamma(\alpha+1)2^{2n+2\alpha-1}}{\Gamma(2n+2\alpha)}$$
$$+\int_{-1}^{1} (1-x^2)^\alpha (Q_n'(x))^2 dx,$$

and noting that

$$\int_{-1}^{1} (1-x^2)^\alpha (P_n(x))^2 dx \geq (a_0^2+a_n^2)\frac{\Gamma(2n+\alpha+1)\Gamma(\alpha+1)2^{2n+2\alpha+1}}{\Gamma(2n+2\alpha+2)}$$
$$+\int_{-1}^{1} (1-x^2)^\alpha (Q_n(x))^2 dx$$

yields

$$\frac{\int_{-1}^{1}(1-x^2)^\alpha (P_n'(x))^2 dx}{\int_{-1}^{1}(1-x^2)^\alpha (P_n(x))^2 dx}$$

(4.3.3)

$$\leq \frac{n^2(a_0^2+a_n^2)\frac{\Gamma(2n+\alpha-1)\Gamma(\alpha+1)2^{2n+2\alpha-1}}{\Gamma(2n+2\alpha)} + \int_{-1}^{1}(1-x^2)^\alpha (Q_n'(x))^2dx}{(a_0^2+a_n^2)\frac{\Gamma(2n+\alpha+1)\Gamma(\alpha+1)2^{2n+2\alpha+1}}{\Gamma(2n+2\alpha+2)} + \int_{-1}^{1}(1-x^2)^\alpha (Q_n(x))^2dx}.$$

Now consider

$$\frac{\int_{-1}^{1}(1-x^2)^\alpha (q_{n0}'(x))^2dx}{\int_{-1}^{1}(1-x^2)^\alpha (q_{n0}(x))^2dx} - \frac{\int_{-1}^{1}(1-x^2)^\alpha (q_{n1}'(x))^2dx}{\int_{-1}^{1}(1-x^2)^\alpha (q_{n1}(x))^2dx}$$

$$= \frac{(n-1)(2n+2\alpha+1)(n+\alpha)f_n(\alpha)}{(\alpha+1)(\alpha+2)(2n+\alpha)(2n+\alpha-1)(2n+\alpha-2)(2n+\alpha-3)},$$

where

$$f_n(\alpha) = 2\alpha^4+(8n-5)\alpha^3+(12n^2-17n+4)\alpha^2+(8n^3-20n^2+11n-1)\alpha-2n(2n^2-5n+4).$$

As the denominator of the above ratio does not change sign, we need only look at $f_n(\alpha)$. It not difficult to check that $f_n(\alpha)$ is increasing on $(-1,\infty)$, and that it has precisely one positive zero, which lies inside $(0,\frac{1}{2})$. We call this zero α_n. From above, we see that for $-1<\alpha\leq\alpha_n$

$$\frac{n^2(2n+2\alpha+1)(n+\alpha)}{2(2n+\alpha)(2n+\alpha-1)} = \frac{\int_{-1}^{1}(1-x^2)^\alpha (q_{n0}'(x))^2dx}{\int_{-1}^{1}(1-x^2)^\alpha (q_{n0}(x))^2dx} \leq \frac{\int_{-1}^{1}(1-x^2)^\alpha (q_{n1}'(x))^2dx}{\int_{-1}^{1}(1-x^2)^\alpha (q_{n1}(x))^2dx}$$

$$= \frac{(2n+2\alpha+1)(n+\alpha)[\alpha(\alpha-1)n^2+2(n-1)(n-(\alpha-1)(2\alpha-1))]}{2(\alpha+1)(\alpha+2)(2n+\alpha-2)(2n+\alpha-3)},$$

and for $\alpha\geq\alpha_n$ the above inequality is reversed. Employing this observation in (4.3.3) completes the proof of Theorem 1. □

For the Proof of Theorem 4.2 we shall use the inequality for $P_n \in L_n$ and $-1 \le x \le 1$

(4.3.4)

$$(1-x^2)((P_n'(x))^2 - P_n(x)P_n''(x)) \le n(P_n(x))^2 - 2xP_n(x)P_n'(x).$$

This inequality is found in the paper of Milovanović and Petković [28], page 284.

Proof of Theorem 4.2

Multiplying (4.3.4) through by $(1-x^2)^2(P_n'(x))^2$ and $(1-x^2)(P_n(x))^2$ we obtain the inequalities

$$(1-x^2)^3(P_n'(x))^2((P_n'(x))^2 - P_n(x)P_n''(x))$$

(4.3.5)

$$\le n(1-x^2)^2(P_n(x))^2(P_n'(x))^2 - 2x(1-x^2)^2P_n(x)(P_n'(x))^3$$

and

(4.3.6)

$$(1-x^2)^2(P_n(x))^2((P_n'(x))^2 - P_n(x)P_n''(x)) \le n(1-x^2)(P_n(x))^4 - 2x(1-x^2)(P_n(x))^3P_n'(x).$$

Denote

$$I_1 = \int_{-1}^{1}(1-x^2)^3(P_n'(x))^4dx.$$

Integrating by parts yields

$$I_1 = -3\int_{-1}^{1}(1-x^2)^3(P_n'(x))^2P_n(x)P_n''(x)dx + 6\int_{-1}^{1}x(1-x^2)^2(P_n'(x))^3P_n(x)dx,$$

and adding $3I_1$ to both sides we obtain

$$4I_1 = 3\int_{-1}^{1}(1-x^2)^3(P_n'(x))^2((P_n'(x))^2 - P_n(x)P_n''(x))dx + 6\int_{-1}^{1}x(1-x^2)^2(P_n'(x))^3P_n(x)dx.$$

Applying (4.3.5) to the above yields the inequality

(4.3.7)

$$4\int_{-1}^{1}(1-x^2)^3(P_n'(x))^4dx \le 3n\int_{-1}^{1}(1-x^2)^2(P_n(x))^2(P_n'(x))^2dx.$$

Now, for any polynomial $P_n(x)$ we have

$$4(1-x^2)^2(P_n(x))^2(P_n'(x))^2 - \frac{d}{dx}\left[(1-x^2)^2(P_n(x))^3P_n'(x)\right]$$
$$= (1-x^2)^2(P_n(x))^2(P_n'(x))^2 + 4x(1-x^2)(P_n(x))^3P_n'(x) - (1-x^2)^2(P_n(x))^3P_n''(x).$$

Thus, integrating both sides of the above from -1 to 1, and then applying (4.3.6) we obtain

$$4\int_{-1}^{1}(1-x^2)^2(P_n(x))^2(P_n'(x))^2dx = \int_{-1}^{1}(1-x^2)^2(P_n(x))^2((P_n'(x))^2 - P_n(x)P_n''(x))dx$$
$$+4\int_{-1}^{1}x(1-x^2)(P_n(x))^3P_n'(x)dx$$

$$\le n\int_{-1}^{1}(1-x^2)(P_n(x))^4dx + 2\int_{-1}^{1}x(1-x^2)(P_n(x))^3P_n'(x)dx.$$

Integrating by parts the last term above yields the inequality

$$4\int_{-1}^{1}(1-x^2)^2(P_n(x))^2(P_n'(x))^2dx \le (n-\frac{3}{2})\int_{-1}^{1}(1-x^2)(P_n(x))^4dx + \int_{-1}^{1}(P_n(x))^4dx.$$

Applying the above to (4.3.7) we get

$$\int_{-1}^{1}(1-x^2)^3(P_n'(x))^4dx \le \frac{3n(2n-3)}{32}\int_{-1}^{1}(1-x^2)(P_n(x))^4dx + \frac{3n}{16}\int_{-1}^{1}(P_n(x))^4dx.$$

Now, replacing n by $2n$ in (4.2.1) with $\alpha = 1$, and then $\alpha = 2$ and $\alpha = 3$ yields

$$\int_{-1}^{1}(1-x^2)^3(P_n'(x))^4dx \le \left(\frac{3n(2n-3)}{32} + \frac{3n(4n+3)(4n+2)}{64(4n+1)}\right)\int_{-1}^{1}(1-x^2)(P_n(x))^4dx$$

$$= \frac{3n^3}{2(4n+1)}\int_{-1}^{1}(1-x^2)(P_n(x))^4dx$$

$$\le \frac{n^3(4n+7)(4n+6)(4n+5)(4n+4)}{64(4n+3)(4n+2)(4n+1)}\int_{-1}^{1}(1-x^2)^3(P_n(x))^4dx,$$

completing the proof of Theorem 4.2. □

CHAPTER 5
TURÁN-TYPE INEQUALITIES

5.1 Main Results

Let H_n denote the set of all polynomials of degree n whose zeros are all real and lie inside $[-1,1]$. Inspired by the inequality of Turán [45], we present three theorems concerned with finding a uniform lower bound for the expression

$$\frac{\int_{-1}^{1}\omega(x)(P_n'(x))^p dx}{\int_{-1}^{1}\omega(x)(P_n(x))^p dx}$$

for $P_n \in H_n$. We provide sharp inequalities for the special cases $p=2$ and $\omega(x) = (1-x^2)^\alpha$, $\alpha > 1$ ($\alpha > -1$ if $P_n(\pm 1) = 0$), and $p=4$, $\omega(x) = (1-x^2)^3$. Then we present an asymptotically sharp result for $\omega(x) \equiv 1$ and p even. Theorem 5.1 generalizes some previous results of A. K. Varma [51],[50], and Theorem 5.2 extends these in a weighted L^4 norm.

Theorem 5.1 Let $P_n \in H_n, n \geq 2$ and $\alpha > 1$ real. Then we have $(n = 2m)$

$$\int_{-1}^{1}(1-x^2)^\alpha (P_n'(x))^2 dx \geq \frac{n^2(2n+2\alpha+1)}{4(n+\alpha-1)(n+\alpha)}\int_{-1}^{1}(1-x^2)^\alpha (P_n(x))^2 dx$$

with equality if and only if $P_n(x) = c(1-x^2)^m$. If $P_n(1) = P_n(-1) = 0$, then the above remains valid for $\alpha > -1$.

Theorem 5.2 Let $P_n \in H_n$. Then we have $(n = 2m)$

$$\int_{-1}^{1}(1-x^2)^3 (P_n'(x))^4 dx \geq \frac{3n^3(4n+7)(4n+5)}{4(4n+6)(4n+4)(4n+2)}\int_{-1}^{1}(1-x^2)^3 (P_n(x))^4 dx$$

with equality if and only if $P_n(x) = c(1-x^2)^m$.

Corollary 5.2.1 Let $P_n \in H_n$. Then we have

(5.1.1)

$$\int_{-1}^{1}(1-x^2)(P_n'(x))^4dx > \frac{3n^3}{8(2n+1)}\int_{-1}^{1}(1-x^2)(P_n(x))^4dx$$

and

(5.1.2)

$$\int_{-1}^{1}(1-x^2)^2(P_n'(x))^4dx > \frac{3n^3(4n+5)}{32(n+1)(2n+1)}\int_{-1}^{1}(1-x^2)^2(P_n(x))^4dx$$

where these results are the best in the sense that there exists a polynomial $P_n^*(x)$ of degree n having all zeros inside $[-1,1]$ ($P_n^*(x) = (1-x^2)^m, n = 2m$) and for which

$$\frac{\int_{-1}^{1}(1-x^2)(P_n^{*\prime}(x))^4dx}{\int_{-1}^{1}(1-x^2)(P_n^*(x))^4dx} = \frac{3n^3(4n+3)(4n+1)}{64(2n+1)(2n-1)(n-1)}$$

and

$$\frac{\int_{-1}^{1}(1-x^2)^2(P_n^{*\prime}(x))^4dx}{\int_{-1}^{1}(1-x^2)^2(P_n^*(x))^4dx} = \frac{3n^3(4n+5)(4n+3)}{64(n+1)(2n+1)(2n-1)}.$$

Theorem 5.3 Let $P_n \in H_n$ and $p \geq 2$ even. Then we have

$$\int_{-1}^{1}(P_n'(x))^pdx >$$

$$\frac{((p-1)n-(p-2))((p-3)n-(p-4))\cdots(5n-4)(3n-2)n(pn+1)}{p^{\frac{p}{2}}(pn-2)}\int_{-1}^{1}(P_n(x))^pdx$$

where this inequality is sharp in the sense that for the polynomial $P_n^*(x) = (1-x^2)^m$ ($n = 2m$) we have

$$\lim_{n\to\infty}\frac{1}{n^{\frac{p}{2}}}\frac{\|P_n^{*\prime}\|_{L^p}^p}{\|P_n^*\|_{L^p}^p} = \frac{(p-1)(p-3)\cdots5\cdot3}{p^{\frac{p}{2}}}.$$

5.2 Some Identities

We shall need the following known identities.

(5.2.1)

$$P_n'(x) = P_n(x)\sum_{k=1}^{n}\frac{1}{x-x_k}$$

(5.2.2)

$$(P_n'(x))^2 - P_n(x)P_n''(x) = (P_n(x))^2\sum_{k=1}^{n}\frac{1}{(x-x_k)^2}$$

(5.2.3)

$$\frac{1-x^2}{(x-x_k)^2} + \frac{2x}{x-x_k} = 1 + \frac{1-x_k^2}{(x-x_k)^2}$$

Identity 5.2.1 Let $\alpha > 0$ and $P_n(x)$ be any algebraic polynomial of degree n with $x_1, x_2, \ldots, x_n$ as its real zeros. Then we have

$$2\int_{-1}^{1}(1-x^2)^{\alpha+1}(P_n'(x))^2dx = (n-(2\alpha+1)\alpha)\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx$$

$$+2\alpha^2\int_{-1}^{1}(1-x^2)^{\alpha-1}(P_n(x))^2dx$$

$$+\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx.$$

Moreover, if $P_n(x)$ vanishes at $x=1$ and $x=-1$, then the above is also valid for $-2 < \alpha \le 0$.

Proof

Integrating by parts, we have for $\alpha > 0$

$$\int_{-1}^{1}(1-x^2)^{\alpha+1}(P_n'(x))^2dx$$

$$= -\int_{-1}^{1}P_n(x)\left[(1-x^2)^{\alpha+1}P_n''(x) - 2(\alpha+1)(1-x^2)^{\alpha}xP_n'(x)\right]dx.$$

Therefore, we obtain

$$2\int_{-1}^{1}(1-x^2)^{\alpha+1}(P_n'(x))^2dx = \int_{-1}^{1}\left[(P_n'(x))^2 - P_n(x)P_n''(x)\right](1-x^2)^{\alpha+1}dx$$

$$+2(1+\alpha)\int_{-1}^{1}(1-x^2)^{\alpha}xP_n(x)P_n'(x)dx.$$

Now, on using (5.2.1)–(5.2.3) we have

$$2\int_{-1}^{1}(1-x^2)^{\alpha+1}(P_n'(x))^2dx = \int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2\sum_{k=1}^{n}\frac{1-x^2}{(x-x_k)^2}dx$$

$$+\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2\sum_{k=1}^{n}\frac{2x}{x-x_k}dx - \alpha\int_{-1}^{1}(P_n(x))^2\{(1-x^2)^{\alpha} - 2\alpha x^2(1-x^2)^{\alpha-1}\}dx$$

$$= \int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx + (n-\alpha)\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx$$

$$-2\alpha^2\int_{-1}^{1}(1-x^2-1)(1-x^2)^{\alpha-1}(P_n(x))^2dx$$

$$= \int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx$$

$$+(n-\alpha(2\alpha+1))\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx$$

$$+2\alpha^2\int_{-1}^{1}(1-x^2)^{\alpha-1}(P_n(x))^2dx.$$

This proves the identity for $\alpha > 0$. For the case $P_n(\pm 1) = 0$ and $-2 < \alpha \leq 0$, the above proof remains valid. Thus, Identity 5.2.1 is established. □

Identity 5.2.2 Let $P_n(x)$ be any algebraic polynomial of degree n with n real zeros $x_1, x_2 \ldots x_n$, and let $\alpha > 0$. Then we have

$$\int_{-1}^{1}(1-x^2)^{\alpha-1}(P_n(x))^2dx = \frac{(2n+2\alpha+1)}{(2n+2\alpha)}\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx$$

$$-\frac{1}{2(n+\alpha)^2}\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx$$

$$+\frac{1}{(n+\alpha)^2}\int_{-1}^{1}(1-x^2)^{\alpha-1}[(1-x^2)P_n'(x)+nxP_n(x)]^2dx.$$

Moreover, if $P_n(1)=0,\ P_n(-1)=0$, then the above is valid for $-2<\alpha\leq 0$ as well.

Proof

First we note that for $\alpha>0$

$$\int_{-1}^{1}(1-x^2)^{\alpha-1}[(1-x^2)P_n'(x)+nxP_n(x)]^2dx=\int_{-1}^{1}(1-x^2)^{\alpha+1}(P_n'(x))^2dx$$

$$+n^2\int_{-1}^{1}(1-x^2)^{\alpha-1}x^2(P_n(x))^2dx+2n\int_{-1}^{1}(1-x^2)^{\alpha}xP_n(x)P_n'(x)dx$$

$$=\int_{-1}^{1}(1-x^2)^{\alpha+1}(P_n'(x))^2dx+n^2\int_{-1}^{1}x^2(1-x^2)^{\alpha-1}(P_n(x))^2dx$$

$$-n\int_{-1}^{1}(P_n(x))^2[(1-x^2)^{\alpha}-2x^2\alpha(1-x^2)^{\alpha-1}]dx$$

$$=\int_{-1}^{1}(1-x^2)^{\alpha+1}(P_n'(x))^2dx+(n^2+2n\alpha)\int_{-1}^{1}x^2(1-x^2)^{\alpha-1}(P_n(x))^2dx$$

$$-n\int_{-1}^{1}(P_n(x))^2(1-x^2)^{\alpha}dx$$

$$=\int_{-1}^{1}(1-x^2)^{\alpha+1}(P_n'(x))^2dx-(n^2+2n\alpha+n)\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx$$

$$+(n^2+2n\alpha)\int_{-1}^{1}(P_n(x))^2(1-x^2)^{\alpha-1}dx$$

$$=\frac{(n-(2\alpha+1)\alpha)}{2}\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx+\alpha^2\int_{-1}^{1}(1-x^2)^{\alpha-1}(P_n(x))^2dx$$

$$+\frac{1}{2}\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx$$

$$-(n^2+2n\alpha+n)\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx$$

$$+(n^2+2n\alpha)\int_{-1}^{1}(P_n(x))^2(1-x^2)^{\alpha-1}dx$$

$$=(n+\alpha)^2\int_{-1}^{1}(P_n(x))^2(1-x^2)^{\alpha-1}dx$$

$$+\left[\frac{n-(2\alpha+1)\alpha}{2}-(n^2+2n\alpha+n)\right]\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx$$

$$+\frac{1}{2}\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx,$$

which is equivalent to the stated identity. We note that if $-2<\alpha$, then the identity is still valid provided $P_n(1)=0$, $P_n(-1)=0$. This proves the Identity 5.2.2. □

Identity 5.2.3 Let P_n be any algebraic polynomial of degree n with $x_1, x_2, \ldots x_n$ its real zeros. Then we have

$$\int_{-1}^{1}(1-x^2)^3(P_n'(x))^4dx=\frac{3n}{4}\int_{-1}^{1}(1-x^2)^2(P_n(x))^2(P_n'(x))^2dx$$

$$+\frac{3}{4}\int_{-1}^{1}(1-x^2)^2(P_n(x))^2(P_n'(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx.$$

Proof

Denote

$$I_1=\int_{-1}^{1}(1-x^2)^3(P_n'(x))^4dx.$$

Integrating by parts yields

$$I_1 = -3\int_{-1}^{1}(1-x^2)^3(P_n'(x))^2P_n(x)P_n''(x)dx + 6\int_{-1}^{1}x(1-x^2)^2(P_n'(x))^3P_n(x)dx$$

and adding $3I_1$ to both sides we obtain

$$4I_1 = 3\int_{-1}^{1}(1-x^2)^3(P_n'(x))^2\left[(P_n'(x))^2 - P_n(x)P_n''(x)\right]dx$$
$$+6\int_{-1}^{1}x(1-x^2)^2(P_n'(x))^3P_n(x)dx.$$

Applying (5.2.1)–(5.2.3) yields the identity. □

Identity 5.2.4. Let P_n be a polynomial of degree n with real zeros $x_1, x_2, \ldots, x_n$. Then for $p \geq 2$ even, we have

$$(p-r-2)\int_{-1}^{1}(P_n'(x))^{p-r}(P_n(x))^r dx + (r+2)\int_{-1}^{1}(1-x^2)(P_n'(x))^{p-r}(P_n(x))^r dx$$
$$= ((p-r-1)n-(p-r-2))\int_{-1}^{1}(P_n'(x))^{p-r-2}(P_n(x))^{r+2}dx$$
$$+(p-r-1)\int_{-1}^{1}(P_n'(x))^{p-r-2}(P_n(x))^{r+2}\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx$$
$$+(p-r-2)\int_{-1}^{1}(P_n'(x))^{p-r-2}(P_n(x))^r(xP_n'(x)-P_n(x))^2dx,$$

$r = 0, 2, \ldots, p-4$.

Proof

Denote

$$I_r = \int_{-1}^{1}(1-x^2)(P_n'(x))^{p-r}(P_n(x))^r dx.$$

Integrating by parts we get

$$I_r = -\frac{p-r-1}{r+1}\int_{-1}^{1}(1-x^2)(P_n'(x))^{p-r-2}(P_n(x))^r P_n(x)P_n''(x)dx$$
$$+\frac{2}{r+1}\int_{-1}^{1}x(P_n'(x))^{p-r-1}(P_n(x))^{r+1}dx$$

and adding $\frac{p-r-1}{r+1}I_r$ to both sides yields

$$\frac{p}{r+1}I_r = \frac{p-r-1}{r+1}\int_{-1}^{1}(1-x^2)(P_n'(x))^{p-r-2}(P_n(x))^r((P_n'(x))^2 - P_n(x)P_n''(x))dx$$
$$+2\frac{p-r-1}{r+1}\int_{-1}^{1}x(P_n'(x))^{p-r-1}(P_n(x))^{r+1}dx$$
$$+\left(\frac{2}{r+1}-\frac{2(p-r-1)}{r+1}\right)\int_{-1}^{1}x(P_n'(x))^{p-r-1}(P_n(x))^{r+1}dx.$$

We now multiply both sides by $r+1$ and apply (5.2.1)-(5.2.3) to obtain

$$pI_r = (p-r-1)n\int_{-1}^{1}(P_n'(x))^{p-r-2}(P_n(x))^{r+2}dx$$
$$+(p-r-1)\int_{-1}^{1}(P_n'(x))^{p-r-2}(P_n(x))^{r+2}\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx$$
$$-2(p-r-1)\int_{-1}^{1}x(P_n'(x))^{p-r-1}(P_n(x))^{r+1}dx$$
$$-(p-r-2)\int_{-1}^{1}(P_n'(x))^{p-r-2}(P_n(x))^r(xP_n'(x)) - P_n(x))^2dx$$
$$+(p-r-2)\int_{-1}^{1}(P_n'(x))^{p-r-2}(P_n(x))^r(xP_n'(x) - P_n(x))^2dx$$

$$= ((p-r-1)n - (p-r-2))\int_{-1}^{1}(P_n'(x))^{p-r-2}(P_n(x))^{r+2}dx$$
$$+(p-r-1)\int_{-1}^{1}(P_n'(x))^{p-r-2}(P_n(x))^{r+2}\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx$$

$$+(p-r-2)\int_{-1}^{1}(P_n'(x))^{p-r-2}(P_n(x))^r(xP_n'(x)-P_n(x))^2dx$$

$$-(p-r-2)\int_{-1}^{1}x^2(P_n'(x))^{p-r}(P_n(x))^r dx,$$

and taking the last term above to the left hand side completes the proof of the identity. □

5.3 Proofs of Theorems

Proof of Theorem 5.1

On using Identities 5.2.1 and 5.2.2, we obtain

$$2\int_{-1}^{1}(1-x^2)^{\alpha+1}(P_n'(x))^2dx=(n-\alpha(2\alpha+1))\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx$$

$$+2\alpha^2\Big[\frac{(2n+2\alpha+1)}{(2n+2\alpha)}\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx$$

$$-\frac{1}{2(n+\alpha)^2}\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx$$

$$+\frac{1}{(n+\alpha)^2}\int_{-1}^{1}(1-x^2)^{\alpha-1}[(1-x^2)P_n'(x)+nxP_n(x)]^2dx\Big]$$

$$+\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx$$

$$=\left[n-(2\alpha+1)\alpha+\frac{\alpha^2(2n+2\alpha+1)}{n+\alpha}\right]\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx$$

$$+\left(1-\frac{\alpha^2}{(n+\alpha)^2}\right)\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx$$

$$+\frac{2\alpha^2}{(n+\alpha)^2}\int_{-1}^{1}(1-x^2)^{\alpha-1}\left[(1-x^2)P_n'(x)+nP_n(x)\right]^2dx$$

$$=\frac{n^2}{n+\alpha}\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2dx+\frac{n^2+2n\alpha}{(n+\alpha)^2}\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx$$

$$+\frac{2\alpha^2}{(n+\alpha)^2}\int_{-1}^{1}(1-x^2)^{\alpha-1}[(1-x^2)P_n'(x)+nxP_n(x)]^2dx$$

$$= \frac{n^2}{n+\alpha}\Big[\frac{(2n+2\alpha+3)}{(2n+2\alpha+2)}\int_{-1}^{1}(1-x^2)^{\alpha+1}(P_n(x))^2dx$$
$$-\frac{1}{2(n+\alpha+1)^2}\int_{-1}^{1}(1-x^2)^{\alpha+1}(P_n(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx$$
$$+\frac{1}{(n+\alpha+1)^2}\int_{-1}^{1}(1-x^2)^{\alpha}[(1-x^2)P_n'(x)+nxP_n(x)]^2dx\Big]$$
$$+\frac{n^2+2n\alpha}{(n+\alpha)^2}\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}$$
$$+\frac{2\alpha^2}{(n+\alpha)^2}\int_{-1}^{1}(1-x^2)^{\alpha-1}[(1-x^2)P_n'(x)+nxP_n(x)]^2dx.$$

Note that

$$-\frac{n^2}{2(n+\alpha)(n+\alpha+1)^2}\int_{-1}^{1}(1-x^2)^{\alpha+1}(P_n(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx$$
$$+\frac{n^2+2n\alpha}{(n+\alpha)^2}\int_{-1}^{1}(1-x^2)^{\alpha}(P_n(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx \geq 0.$$

Therefore, we have

$$2\int_{-1}^{1}(1-x^2)^{\alpha+1}(P_n'(x))^2dx \geq \frac{n^2(2n+2\alpha+3)}{(n+\alpha)(2n+2\alpha+2)}\int_{-1}^{1}(1-x^2)^{\alpha+1}(P_n(x))^2dx,$$

proving the theorem. □

Proof of Theorem 5.2

Let $P_n(x)$ be any algebraic polynomial of degree n with $x_1, x_2, \ldots x_n$ as its real zeros. Then we have

$$4(1-x^2)^2(P_n(x))^2(P_n'(x))^2 - \frac{d}{dx}[(1-x^2)^2(P_n(x))^3P_n'(x)]$$

$$= (1-x^2)^2(P_n(x))^2(P_n'(x))^2 + 4x(1-x^2)(P_n(x))^3P_n'(x) - (1-x^2)^2(P_n(x))^3P_n''(x).$$

We integrate both sides from -1 to 1, and make use of (5.2.1)–(5.2.3). We then obtain

(5.3.1)

$$4\int_{-1}^{1}(1-x^2)^2(P_n(x))^2(P_n'(x))^2dx = \int_{-1}^{1}(1-x^2)^2(P_n(x))^2(P_n'(x))^2 - P_n(x)P_n''(x))dx$$
$$+4\int_{-1}^{1}x(1-x^2)(P_n(x))^3P_n'(x)dx$$

$$= \int_{-1}^{1}(1-x^2)(P_n(x))^4\sum_{k=1}^{n}\frac{1-x^2+2x(x-x_k)}{(x-x_k)^2}dx + 2\int_{-1}^{1}x(1-x^2)(P_n(x))^3P_n'(x)dx$$

$$= \int_{-1}^{1}(1-x^2)(P_n(x))^4\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx + n\int_{-1}^{1}(1-x^2)(P_n(x))^4dx$$
$$-\frac{1}{2}\int_{-1}^{1}(1-3x^2)(P_n(x))^4dx$$

$$= \int_{-1}^{1}(1-x^2)(P_n(x))^4\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx + (n-\frac{3}{2})\int_{-1}^{1}(1-x^2)(P_n(x))^4dx + \int_{-1}^{1}(P_n(x))^4dx.$$

From Identity 5.2.3 we have

$$\int_{-1}^{1}(1-x^2)^3(P_n'(x))^4dx = \frac{3n}{4}\int_{-1}^{1}(1-x^2)^2(P_n(x))^2(P_n'(x))^2dx$$
$$+\frac{3}{4}\int_{-1}^{1}(1-x^2)^2(P_n(x))^2(P_n'(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx.$$

Now, using (5.3.1) we obtain

(5.3.2)

$$\int_{-1}^{1}(1-x^2)^3(P_n'(x))^4dx = \frac{3n}{16}\Big[\left(n-\frac{3}{2}\right)\int_{-1}^{1}(1-x^2)(P_n(x))^4dx + \int_{-1}^{1}(P_n(x))^4dx$$

$$+\int_{-1}^{1}(1-x^2)(P_n(x))^4\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx\Big]$$

$$+\frac{3}{4}\int_{-1}^{1}(1-x^2)^2(P_n(x))^2(P_n'(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx.$$

Also, from Identity 5.2.2 with $\alpha = 1$ and n replaced by $2n$, and on noting that in this case

$$\sum_{k=1}^{2n}\frac{1-x_k^2}{(x-x_k)^2} = 2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2},$$

we have

$$\int_{-1}^{1}(P_n(x))^4dx \geq \frac{(4n+3)}{2(2n+1)}\int_{-1}^{1}(1-x^2)(P_n(x))^4dx$$

$$-\frac{1}{(2n+1)^2}\int_{-1}^{1}(1-x^2)(P_n(x))^4\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx.$$

Therefore, on using (5.3.2) and the above we have

(5.3.3)

$$\int_{-1}^{1}(1-x^2)^3(P_n'(x))^4dx \geq \frac{3n}{16}\Big[\left(n-\frac{3}{2}\right)\int_{-1}^{1}(1-x^2)(P_n(x))^4dx$$

$$+\frac{4n+3}{2(2n+1)}\int_{-1}^{1}(1-x^2)(P_n(x))^4dx$$

$$+\left(1-\frac{1}{(2n+1)^2}\right)\int_{-1}^{1}(1-x^2)(P_n(x))^4\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx\Big]$$

$$+\frac{3}{4}\int_{-1}^{1}(1-x^2)^2(P_n(x))^2(P_n'(x))^2\sum_{k=1}^{n}\frac{(1-x_k^2)}{(x-x_k)^2}dx$$

$$=\frac{3n}{16}\Big[\frac{2n^2}{2n+1}\int_{-1}^{1}(1-x^2)(P_n(x))^4dx+\frac{4n(n+1)}{(2n+1)^2}\int_{-1}^{1}(1-x^2)(P_n(x))^4\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx\Big]$$

$$+\frac{3}{4}\int_{-1}^{1}(1-x^2)^2(P_n(x))^2(P_n'(x))^2\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx.$$

Again, using Identity 5.2.2 with n replaced by $2n$ and $\alpha = 2$ we have

(5.3.4)
$$\int_{-1}^{1}(1-x^2)(P_n(x))^4dx \geq \frac{(4n+5)}{(4n+4)}\int_{-1}^{1}(1-x^2)^2(P_n(x))^4dx$$
$$-\frac{1}{(2n+2)^2}\int_{-1}^{1}(1-x^2)^2(P_n(x))^4\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx,$$

and similarly with $\alpha = 3$

(5.3.5)
$$\int_{-1}^{1}(1-x^2)^2(P_n(x))^4dx \geq \frac{(4n+7)}{(4n+6)}\int_{-1}^{1}(1-x^2)^3(P_n(x))^4dx$$
$$-\frac{1}{(2n+3)^2}\int_{-1}^{1}(1-x^2)^3(P_n(x))^4\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx.$$

Therefore, we obtain

$$\int_{-1}^{1}(1-x^2)(P_n(x))^4dx \geq \frac{4n+5}{4n+4}\Big[\frac{4n+7}{4n+6}\int_{-1}^{1}(1-x^2)^3(P_n(x))^4dx$$
$$-\frac{1}{(2n+3)^2}\int_{-1}^{1}(1-x^2)^3(P_n(x))^4\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx\Big]$$
$$-\frac{1}{(2n+2)^2}\int_{-1}^{1}(1-x^2)^2(P_n(x))^4\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx$$

$$=\frac{(4n+5)(4n+7)}{8(n+1)(2n+3)}\int_{-1}^{1}(1-x^2)^3(P_n(x))^4dx$$
$$-\frac{(4n+5)}{4(n+1)(2n+3)^2}\int_{-1}^{1}(1-x^2)^3(P_n(x))^4\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx$$
$$-\frac{1}{(2n+2)^2}\int_{-1}^{1}(1-x^2)^2(P_n(x))^4\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx$$

$$\geq\frac{(4n+5)(4n+7)}{8(n+1)(2n+3)}\int_{-1}^{1}(1-x^2)^3(P_n(x))^4dx$$

$$-\left(\frac{(4n+5)}{4(n+1)(2n+3)^2}+\frac{1}{(2n+2)^2}\right)\int_{-1}^{1}(1-x^2)(P_n(x))^4\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx.$$

Thus,

$$\frac{2n^2}{2n+1}\int_{-1}^{1}(1-x^2)(P_n(x))^4dx+\frac{4n(n+1)}{(2n+1)^2}\int_{-1}^{1}(1-x^2)(P_n(x))^4\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx$$

$$\begin{aligned}&\geq\frac{2n^2}{(2n+1)}\frac{(4n+5)(4n+7)}{8(n+1)(2n+3)}\int_{-1}^{1}(1-x^2)^3(P_n(x))^4dx\\&+\left[\frac{4n(n+1)}{(2n+1)^2}-\frac{2n^2}{(2n+1)}\left(\frac{(4n+5)}{4(n+1)(2n+3)^2}+\frac{1}{(2n+2)^2}\right)\right]\\&\cdot\int_{-1}^{1}(1-x^2)(P_n(x))^4\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}dx,\end{aligned}$$

and from (5.3.3) we deduce

$$\int_{-1}^{1}(1-x^2)^3(P_n'(x))^4dx\geq\frac{3n^3(4n+5)(4n+7)}{64(2n+1)(n+1)(2n+3)}\int_{-1}^{1}(1-x^2)^3(P_n(x))^4dx.$$

This proves Theorem 5.2. Now, as

$$\int_{-1}^{1}(1-x^2)(P_n'(x))^4dx>\int_{-1}^{1}(1-x^2)^2(P_n'(x))^4dx>\int_{-1}^{1}(1-x^2)^3(P_n'(x))^4dx$$

we deduce (5.1.1) from (5.3.3). Similarly, we deduce (5.1.2) from (5.3.4) and (5.3.5). This proves the Corollary. □

Proof of Theorem 5.3

Let $P_n\in H_n$ and $p\geq 2$ even. Denote

$$J_r=\int_{-1}^{1}(P_n'(x))^{p-r}(P_n(x))^r dx.$$

From Identity 5.2.4 we see that

$$pJ_r > ((p-r-1)n-(p-r-2))J_{r+2} \quad (r=0,2,\dots,p-4)$$

which yields

$$\begin{aligned}
\frac{\int_{-1}^{1}(P_n'(x))^p dx}{\int_{-1}^{1}(P_n(x))^p dx} &= \frac{J_0}{J_p} = \left(\frac{J_0}{J_2}\frac{J_2}{J_4}\cdots\frac{J_{p-4}}{J_{p-2}}\right)\cdot\frac{J_{p-2}}{J_p} \\
&> \left(\frac{(p-1)n-(p-2)}{p}\right)\left(\frac{(p-3)n-(p-4)}{p}\right)\cdots\left(\frac{3n-2}{p}\right) \\
&\quad \cdot\frac{\int_{-1}^{1}(P_n(x))^{p-2}(P_n'(x))^2 dx}{\int_{-1}^{1}(P_n(x))^p dx}
\end{aligned}$$

$$= \frac{((p-1)n-(p-2))((p-3)n-(p-4))\dots(3n-2)}{p^{\frac{p}{2}-1}}\left(\frac{2}{p}\right)^2\frac{\int_{-1}^{1}\left(\frac{d}{dx}(P_n^{\frac{p}{2}}(x))\right)^2 dx}{\int_{-1}^{1}\left(P_n^{\frac{p}{2}}(x)\right)^2 dx}$$

$$\geq \frac{((p-1)n-(p-2))((p-3)n-(p-4))\dots(3n-2)4}{p^{\frac{p}{2}+1}}\times\frac{pn(pn+1)}{4(pn-2)}$$

$$= \frac{((p-1)n-(p-2))((p-3)n-(p-4))\dots(3n-2)n(pn+1)}{p^{\frac{p}{2}}(pn-2)},$$

with the last inequality following from Theorem 1.4.4. Now, we use that

$$\int_0^1 t^{p-1}(1-t)^{q-1}dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

where p and q are positive real numbers. To see that the result above is asymptotically sharp, we note that for the polynomial $P_0(x)=(1-x^2)^m$ $(n=2m)$ we have

$$\frac{\int_{-1}^{1}(P_0'(x))^p dx}{\int_{-1}^{1}(P_0(x))^p dx} = \frac{n^p\int_{-1}^{1}x^p(1-x^2)^{\frac{(n-2)p}{2}}dx}{\int_{-1}^{1}(1-x^2)^{\frac{np}{2}}dx}$$

$$= \frac{n^p \int_0^1 t^{\frac{p-1}{2}} (1-t)^{\frac{(n-2)p}{2}} dt}{\int_0^1 t^{-\frac{1}{2}} (1-t)^{\frac{np}{2}} dt}$$

$$= \frac{n^p \left(\frac{p-1}{2}\right) \left(\frac{p-3}{2}\right) \cdots \left(\frac{3}{2}\right) \left(\frac{1}{2}\right) \Gamma\left(\frac{np-(2p-2)}{2}\right) \Gamma\left(\frac{np+3}{2}\right)}{\Gamma\left(\frac{np-(p-3)}{2}\right) \Gamma\left(\frac{np+2}{2}\right)}$$

$$= \frac{n^p \frac{(p-1)(p-3)\ldots3\cdot1}{2^{\frac{p}{2}}} \frac{(np+1)(n-p-1)\ldots(np-(p-3))}{2^{\frac{p}{2}}}}{\frac{np(np-2)\ldots(np-(2p-2))}{2^p}}$$

$$= n^p (p-1)(p-3)\ldots5\cdot3 \, \frac{(np+1)(np-1)\cdots(np-(p-3))}{np(np-2)\cdots(np-(2p-2))}.$$

This proves Theorem 5.3. $\square$

CHAPTER 6
SUMMARY AND CONCLUSIONS

6.1 Synopsis

The problems discussed in this dissertation come from the vast field of Approximation Theory. The areas of this field considered here are Birkhoff Interpolation, Erdös-type inequalities and Turán-type inequalities.

In Chapter Two, we proved the existence and uniqueness in the case of the 'modified' $(0,1,3,4)$ case of Birkhoff Interpolation on the zeros of $\pi_n(x)$, and then provided an explicit representation in this case. Chapter Three presents the estimates of the fundamental polynomials, which are then used to prove the uniform convergence of the 'modified' $(0,1,3,4)$ Birkhoff-Fejér operator for the entire class of continuous functions on the interval $[-1,1]$. This provides only the second known case of such a Birkhoff-Fejér operator.

In Chapter Four, a classical theorem of P. Erdös [10] is extended in the L^2 norm with the ultraspherical weight $\omega(x) = (1-x^2)^\alpha, \alpha > -1$, for the Lorentz class L_n of polynomials. This result also generalizes some previously known partial results in the L^2 norm. Then the result of Erdös is extended in the L^4 norm with weight $\omega(x) = (1-x^2)^3$. These results are the best possible, as the inequalities given are sharp. We note that this is the first sharp result extending the inequality of Erdös in the L^4 norm.

In Chapter Five, a classical theorem of P. Turán [45] is extended in the L^2 norm with ultraspherical weight $\omega(x) = (1-x^2)^\alpha, \alpha > 1$ ($\alpha > -1$ if $P_n(\pm 1) = 0$). Next, we extended the inequality of Turán in the L^4 norm with weight $\omega(x) = (1-x^2)^3$. We note that this is the first sharp extension of the inequality of Turán in the L^4 norm.

These results provide lower bound analogues to the results given in Chapter Four for the class H_n of polynomials. These results are again the best possible, providing inequalities that are sharp. Finally, the classical theorems of P. Turan (in the uniform norm) and A.K. Varma [51] (in the L^2 norm) are extended in the L^p norm for p an even integer. This result is sharp in the asymptotic sense.

6.2 Open Problems

We close this chapter by noting some open problems related to those presented in this work. First, we recall that the case of the 'modified' $(0,1,3,4)$ interpolation has been generalized into the 'modified' $(0,\ldots,r-3,r-1,r)$ case. Similarly, the 'modified' (0,2), and the 'modified' (0,3) and (0,1,4) cases have been extended to the 'modified' $(0,\ldots,r-2,r)$ and $(0,\ldots,r-3,r)$ cases, respectively. In the paper of A. Sharma, J. Szabados, A.K. Varma, and the author [38], the problem of existence and uniqueness is settled, and explicit forms are provided. We note that the explicit representation of the last fundamental polynomials is found in the same manner as that given here. However, the remaining fundamental polynomials are determined in a different manner, one that is more amenable to the general situation. Nevertheless, better estimates are obtained from those given in this work. One expects that the $(0,3)$ and $(0,1,3,4)$ cases are the only two of all the previously studied cases which converge uniformly for the entire class $C[-1,1]$. These two cases have a special 'balanced' nature.

One problem would be to show that the given error of $O\left(\omega_1(f,\frac{\log n}{n})\right)$ is sharp, and if not, to provide a sharper estimate. While it is easy to write down an explicit representation in the 'pure' $(0,1,3,4)$ case, having handled the 'modified' case, this representation is too complicated to be useful as a means of estimating the fundamental polynomials in the 'pure' case. Thus, convergence results cannot be determined. Hence, another problem would be to find an even simpler explicit representation in the 'pure' (or 'modified') $(0,1,3,4)$ case, and then provide convergence results in the

'pure' case. One would expect the error to be $O\Big(\omega_4(f, \frac{\log n}{n})\Big)$. Another problem would be to find other Birkhoff-Fejér operators which also converge uniformly for all continuous functions on the interval $[-1, 1]$.

Concerning Chapter Four, it would be interesting to provide the sharp inequality in the L^4 norm when $\omega(x) \equiv 1$, and in the more general case $\omega(x) = (1 - x^2)^\alpha$, $\alpha > -1$, as well as providing sharp results in the L^p norm for larger values of p. Also interesting is the extension of these results to higher derivatives. For the related extensions already known in the uniform norm, see W.A. Markov [25] (for all higher derivatives of algebraic polynomials of degree n) and J.T. Scheick [36] (for first and second derivatives of polynomials in the Lorentz class).

In Chapter Five, again it would be interesting to provide the sharp inequality in the L^4 norm when $\omega(x) \equiv 1$, as well as when $\omega(x)$ is the ultraspherical weight. Also, it would be interesting to provide the sharp inequality in the L^p norm, p even, for each n, and also for higher derivatives. For the known results in the uniform norm for the class H_n, see J. Eröd [12] (for the sharp result in the case of the first derivative) and V.F. Babenko and S.A. Pichugov [4] (for the case of the second derivative).

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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